35. CROSS-SECTION FORMULAE FOR SPECIFIC PROCESSES

35.1. Leptoproduction


\begin{align*}
\text{Figure 35.1: Kinematic quantities for description of lepton-nucleon scattering. } & \text{ } k \text{ and } k' \text{ are the four-momenta of incoming and outgoing leptons, } P \text{ is the four-momentum of a nucleon with mass } M. \text{ The exchanged particle is a } \gamma, W^\pm, \text{ or } Z^0; \text{ it transfers four-momentum } q = k - k' \text{ to the target.}
\end{align*}

Invariant quantities:

\[ \nu = \frac{q \cdot P}{E} = E - E' \text{ is the lepton's energy loss in the lab (in earlier literature sometimes } \nu = q \cdot P). \text{ Here, } E \text{ and } E' \text{ are the initial and final lepton energies in the lab.} \]

\[ Q^2 = -q^2 = 2(EE' - k \cdot k') - m_1^2 - m_2^2 \text{ where } m_i(m_{1,2}) \text{ is the initial (final) lepton mass. If } EE' \sin^2(\theta/2) \gg m_1^2, m_2^2, \text{ then} \]

\[ \approx 4EE' \sin^2(\theta/2), \text{ where } \theta \text{ is the lepton's scattering angle in the lab.} \]

\[ x = \frac{Q^2}{2M \nu} \text{ In the parton model, } x \text{ is the fraction of the target nucleon's momentum carried by the struck quark. [See section on Quantum Chromodynamics (Sec. 9 of this review.)]} \]

\[ y = \frac{q \cdot P}{k \cdot P} = \frac{\nu}{E} \text{ is the fraction of the lepton's energy lost in the lab.} \]

\[ W^2 = (P + q)^2 = M^2 + 2M \nu - Q^2 \text{ is the mass squared of the system recoiling against the lepton.} \]

35.1.1. Leptoproduction cross sections:

\[ \frac{d^2 \sigma}{dx \ dy} = \nu (s - M^2) \frac{d^2 \sigma}{dx \ dq^2} \frac{2 \pi M \nu}{E E' \sin^2(\theta/2) M_{\text{lab}} \sin(\theta/2)} \]

\[ = x(s - M^2) \frac{d^2 \sigma}{dx \ dq^2}. \tag{35.1} \]

35.1.2. Leptoproduction structure functions: The neutral-current process, } e^+N \rightarrow e^+X, \text{ at low } Q^2 \text{ is just electromagnetic and parity conserving. It can be written in terms of two structure functions } F_1^{em}(x, Q^2) \text{ and } F_2^{em}(x, Q^2); \]

\[ \frac{d^2 \sigma}{dx \ dq^2} = 4\pi \alpha^2(s - M^2) \frac{x F_2^{em}}{Q^2} \left(1 - y - \frac{1}{x} \frac{M^2 y}{(s - M^2)} + y^2 \frac{1}{x} \frac{M^2 y}{(s - M^2)} \right) \]  

\[ \times \left[1 - y - \frac{M^2 y}{(s - M^2)} \right] \tag{35.2} \]

The charged-current processes, } e^-N \rightarrow \nu X, \nu N \rightarrow e^-X, \text{ and } } \tau \nu \rightarrow \nu X, \text{ are parity violating and can be written in terms of three structure functions } F_1^{CC}(x, Q^2), F_2^{CC}(x, Q^2), \text{ and } F_3^{CC}(x, Q^2); \]

\[ \frac{d^2 \sigma}{dx \ dq^2} = G_F^2 \left(1 - M^2 / (s - M^2) \right) \frac{M_W^2}{(Q^2 + M_W^2)^2} \]

\[ \times \left[1 - y - \frac{M^2 y}{(s - M^2)} \right] \tag{35.3} \]

\[ \text{where the last term is positive for the } e^- \text{ and } \nu \text{ reactions and negative for } \tau \nu \rightarrow e^-X. \text{ As explained below there are different structure functions for charge-raising and charge-lowering currents.} \]

35.1.3. Structure functions in the QCD parton model: In the QCD parton model, the structure functions defined above can be expressed in terms of parton distribution functions. The quantity } f_i(x, Q^2) \text{ is the probability that a parton of type } i \text{ (quark, antiquark, or gluon), carries a momentum fraction between } x \text{ and } x + dx \text{ of the nucleon's momentum in a frame where the nucleon's momentum is large. For the cross section corresponding to the neutral-current process } e^+p \rightarrow X, \text{ we have for } s \gg M^2 \text{ (in the case where the incoming electron is either left- (L) or right- (R) handed):} \]

\[ \frac{d^2 \sigma}{dx \ dq^2} = \frac{\alpha^2}{8\pi^2 y^2} \left[ \sum_q \left( x f_q(x, Q^2) + x f_{\bar q}(x, Q^2) \right) \right] \]

\[ \times \left[ A_q + (1 - y)^2 B_q \right]. \tag{35.4} \]

\[ \text{Here the index } q \text{ refers to a quark flavor (i.e., } u, d, s, c, b, \text{ or } t), \text{ and} \]

\[ A_q = \left( -q_x + g_{Lq} g_{Le} \frac{Q^2}{Q^2 + M_Z^2} \right)^2 + \left( -q_x + g_{Rq} g_{Re} \frac{Q^2}{Q^2 + M_Z^2} \right)^2 \]

\[ B_q = \left( -q_x + g_{Rq} g_{Le} \frac{Q^2}{Q^2 + M_Z^2} \right)^2 + \left( -q_x + g_{Lq} g_{Re} \frac{Q^2}{Q^2 + M_Z^2} \right)^2 \]

\[ \text{Here } q_x \text{ is the charge of flavor } q. \text{ For a left-handed electron, } g_{Le} = 0 \text{ and } g_{Lq} = (-1/2 + sin^2 \theta_W)/(sin \theta_W cos \theta_W), \text{ while for a right-handed electron, } g_{Re} = 0 \text{ and } g_{Rq} = (sin^2 \theta_W)/(sin \theta_W cos \theta_W). \]

\[ \text{For the quarks, } g_{Lq} = (T_3 - q_x sin^2 \theta_W)/(sin \theta_W cos \theta_W), \text{ and } B_{Rq} = (-q_x sin^2 \theta_W)/(sin \theta_W cos \theta_W). \]

\[ \text{For neutral-current neutrino (antineutrino) scattering, the same formula applies with } g_{Le} \text{ replaced by } g_{L\bar{\nu}} = 1/(2 sin \theta_W cos \theta_W) \text{ and } g_{Re} \text{ replaced by } g_{R\bar{\nu}} = 0 \text{ [g}_{\bar{\nu}\tau} = -1/(2 sin \theta_W cos \theta_W)].} \]

In the case of the charged-current processes } e^+p \rightarrow \nu X \text{ and } \tau \nu p \rightarrow e^+X, \text{ Eq. (35.3) applies with} \]

\[ F_2 = 2x F_1 = 2x \left[ f_d(x, Q^2) + f_s(x, Q^2) + f_t(x, Q^2) \right] + f_{\bar{\nu} X}(x, Q^2) + f_{\bar{\nu} X}(x, Q^2), \tag{35.7} \]

\[ F_3 = 2f_d(x, Q^2) + f_s(x, Q^2) + f_t(x, Q^2), \tag{35.8} \]

\[ - f_{\bar{\nu} X}(x, Q^2) - f_{\bar{\nu} X}(x, Q^2). \]

\[ \text{For the process } e^-p \rightarrow e^-X: \]

\[ F_2 = 2x F_1 = 2x \left[ f_d(x, Q^2) + f_s(x, Q^2) + f_t(x, Q^2) \right] + f_{\bar{\nu} X}(x, Q^2) + f_{\bar{\nu} X}(x, Q^2), \tag{35.9} \]

\[ F_3 = 2f_d(x, Q^2) + f_s(x, Q^2) + f_t(x, Q^2), \tag{35.10} \]

\[ - f_{\bar{\nu} X}(x, Q^2) - f_{\bar{\nu} X}(x, Q^2). \]

35.2. } e^+e^- \text{ annihilation: For pointlike, spin-1/2 fermions, the differential cross section in the c.m. for } e^+e^- \rightarrow f \bar{f} \text{ via single photon annihilation is } (\theta \text{ is the angle between the incident electron and the produced fermion; } N_e = 1 \text{ if } f \text{ is a lepton and } N_e = 3 \text{ if } f \text{ is a quark).} \]

\[ \frac{d \sigma}{d \Omega} = N_e \frac{\alpha^2}{4 \pi} \beta \left(1 + \cos^2 \theta + (1 - \beta^2) sin^2 \theta \right) Q_f^2, \tag{35.11} \]

\[ \text{where } \beta \text{ is the velocity of the final state fermion in the c.m. and } Q_f \text{ is the charge of the fermion in units of the proton charge. For } \beta \rightarrow 1, \]

\[ \sigma = N_e \frac{4 \pi \alpha^2}{3 \alpha} Q_f^2 = N_e \frac{86.8 Q_f^2 n b}{s (GeV/c^2)^2}. \tag{35.12} \]
At higher energies, the $Z^0$ (mass $M_Z$ and width $\Gamma_Z$) must be included. If the mass of a fermion $f$ is much less than the mass of the $Z^0$, then the differential cross section for $e^+e^- \to f\bar{f}$ is

$$\frac{d^3\sigma}{d^3p} = N\frac{\alpha^2}{4\pi} \left\{ (1 + \cos^2 \theta) \left[ Q_f^2 - 2\chi q_e q_f Q_f + \chi^2 (\alpha^2_e + v_e^2)(\alpha^2_f + v_f^2) \right] + 2\cos \theta \left[ -2\chi q_e q_f J_f + 4\chi_2 q_e q_f \alpha q_f \right] \right\}$$

where

$$\chi_1 = \frac{1}{16\sin^2 \theta_W \cos \theta_W} (s - M_Z^2 + M_Z^2 \Gamma_Z)$$

$$\chi_2 = \frac{1}{256\sin^2 \theta_W \cos^3 \theta_W} (s - M_Z^2)^2 + M_Z^2 \Gamma_Z^2$$

$$a_1 = -1$$

$$v_e = -1 + 4\sin^2 \theta_W$$

$$a_f = 2T_{3f}$$

$$v_f = 2T_{3f} - 4Q_f \sin^2 \theta_W$$ (3.14)

$T_{3f}$ is the third component of the weak isospin of the quark or lepton, $\Gamma_Z$ is the width of the $Z^0$, while $T_{3f}$ is $-1/2$ for $u$, $c$ and neutrinos, and $T_{3f}$ is $1/2$ for $d$, $s$, $b$, and negatively charged leptons.

At LEP II it may be possible to produce the orthodox Higgs boson, $H$, (see the mini-review on Higgs bosons) in the reaction $e^+e^- \to HZ^0$, which proceeds dominantly through a virtual $Z^0$. The Standard Model prediction for the cross section is

$$\sigma(e^+e^-\to HZ^0) = \frac{\sigma^2}{24\sqrt{3}} K^2 \frac{K^2 + 3M_H^2}{(s - M_Z^2)^2} \left[ 1 - 4\sin^2 \theta_W + 8\sin^4 \theta_W \right]$$

where $K$ is the c.m. momentum of the produced $H$ or $Z^0$. Near the production threshold, this formula needs to be corrected for the finite width of the $Z^0$.

### 3.5.3. Two-photon process at $e^+e^-$ colliders

When an $e^+$ and an $e^-$ collide with energies $E_1$ and $E_2$, they emit $\gamma\gamma$ virtual photons with energies $\omega_1$ and $\omega_2$ and 4-momenta $q_1$ and $q_2$. In the equivalent photon approximation, the cross section for $e^+e^-\to e^+e^-X$ is related to the cross section for $\gamma\gamma \to X$ by (Ref. 1)

$$\frac{d^2\sigma}{d^2p} = \frac{\alpha^2}{36\pi} \int_{\gamma\gamma} \left( \frac{f(z)}{z} \right) \left[ \left( \frac{\Gamma_{\gamma\gamma} \gamma\gamma}{\Gamma_{\gamma\gamma}} \right) - \left( \frac{\Gamma_{\gamma\gamma} \gamma\gamma}{\Gamma_{\gamma\gamma}} \right) \right]$$

where $\Gamma_{\gamma\gamma}$ is the width of the $\gamma\gamma$ resonance, while $\Gamma_{\gamma\gamma}$ is the width of the $\gamma\gamma$ resonance.

### 3.5.4. Inclusive hadronic reactions

One-particle inclusive cross sections $E d^2\sigma/d^2p$ for the production of particles of momentum $p$ are conveniently expressed in terms of rapidity (see above) and the momentum $p_y$ transverse to the beam direction (defined in the center-of-mass frame)

$$E \frac{d^2\sigma}{d^2p} = \frac{d^3\sigma}{d\phi dy dp_y}$$

In the case of processes where $p_y$ is large or the mass of the produced particle is large (here large means greater than 10 GeV), the parton model can be used to calculate the rate. Symbolically

$$\sigma_{\text{hadronic}} = \sum_{ij} f_i(x_1, Q^2) f_j(x_2, Q^2) dx_1 dx_2 \frac{d^2\sigma}{d^2p} = \sum_{ij} f_i(x_1, Q^2) f_j(x_2, Q^2) dx_1 dx_2 \frac{d^2\sigma}{d^2p}$$

where $x_1 = \sqrt{s} e^{-y_1}$ and $x_2 = \sqrt{s} e^{-y_2}$ and $y = M^2_{ij}/x$. Similarly the production of a jet in $pp$ (or $p\bar{p}$) collisions is given by

$$\frac{d^2\sigma}{d^2p} = \frac{\alpha^2}{36\pi} \int_{\gamma\gamma} \left( \frac{f(z)}{z} \right) \left[ \left( \frac{\Gamma_{\gamma\gamma} \gamma\gamma}{\Gamma_{\gamma\gamma}} \right) - \left( \frac{\Gamma_{\gamma\gamma} \gamma\gamma}{\Gamma_{\gamma\gamma}} \right) \right]$$

where the summation is over quarks, gluons, and antiquarks. Here

$$s = (p_1 + p_2)^2$$

$$t = (p_1 - p_2)^2$$

$$u = (p_2 - p_2)^2$$

$p_1$ and $p_2$ are the momenta of the incoming $p$ and $p$ (or $\bar{p}$) and $\hat{s}$, $\hat{t}$, and $\hat{u}$ are $s$, $t$, and $u$ with $p_1 \to x_1 p_1$ and $p_2 \to x_2 p_2$. The partonic cross section $\frac{d^2\sigma}{d^2p}$ can be found in Ref. 2. Example: for the process $gg \to q\bar{q}$

$$\frac{d^2\sigma}{d^2p} = 3\alpha^2 \int_{\gamma\gamma} \left( \frac{f^2 + g^2}{f^2 + g^2} \right) \left[ \frac{4}{9\hat{s} - 1} \right]$$

The prediction of Eq. (3.23) is compared to data from the UA1 and UA2 collaborations in Fig. 37.8 in the Plots of Cross Sections and Related Quantities section of this Review.

The associated production of a Higgs boson and a gauge boson is analogous to the process $e^+e^-\to HZ^0$ in Sec. 3.5.2. The required parton-level cross sections (Ref. 34), averaged over initial quark colors, are

$$\sigma(q\bar{q} \to W^+H) = \frac{\alpha^2 V_{ij}}{36\sin^2 \theta_W} \left( \frac{K^2 + 3M_H^2}{(s - M_W^2)^2} \right)$$

$$\sigma(q\bar{q} \to Z^0H) = \frac{\alpha^2 (\alpha^2 + v_H^2)}{144\sin^4 \theta_W \cos^2 \theta_W} \left( \frac{K^2 + 3M_H^2}{(s - M_Z^2)^2} \right)$$

Here $V_{ij}$ is the appropriate element of the Kobayashi-Maskawa matrix and $K$ is the c.m. momentum of the produced $H$. The axial and vector couplings are defined as in Sec. 3.5.2.
35. One-particle inclusive distributions

In order to describe one-particle inclusive production in $e^+e^-$ annihilation or deep inelastic scattering, it is convenient to introduce a fragmentation function $D^h_i(z,Q^2)$ where $D^h_i(z,Q^2)$ is the number of hadrons of type $h$ and momentum between $zp$ and $(z + dz)p$ produced in the fragmentation of a parton of type $i$. The $Q^2$ evolution is predicted by QCD and is similar to that of the parton distribution functions [see section on Quantum Chromodynamics (Sec. 9 of this Review)]. The $D^h_i(z,Q^2)$ are normalized so that

$$\sum_h \int zD^h_i(z,Q^2)dz = 1.$$  \hspace{1cm} (35.28)$$

If the contributions of the $Z$ boson and three-jet events are neglected, the cross section for producing a hadron $h$ in $e^+e^-$ annihilation is given by

$$\frac{1}{\sigma_{\text{had}}} \frac{d\sigma}{dz} = \frac{\sum_i e^2_i D^h_i(z,Q^2)}{\sum_i e^2_i},$$  \hspace{1cm} (35.29)$$

where $e_i$ is the charge of quark-type $i$, $\sigma_{\text{had}}$ is the total hadronic cross section, and the momentum of the hadron is $zE_{\text{cm}}/2$.

In the case of deep inelastic muon scattering, the cross section for producing a hadron of energy $E_h$ is given by

$$\frac{1}{\sigma_{\text{tot}}} \frac{d\sigma}{dz} = \frac{\sum_i e^2_i q_i(x,Q^2) D^h_i(z,Q^2)}{\sum_i e^2_i q_i(x,Q^2)},$$  \hspace{1cm} (35.30)$$

where $E_h = \nu z$. (For the kinematics of deep inelastic scattering, see Sec. 34.4.2 of the Kinematics section of this Review.) The fragmentation functions for light and heavy quarks have a different $z$ dependence; the former peak near $z = 0$. They are illustrated in Figs. 36.1 and 36.2 in the section on “Heavy Quark Fragmentation in $e^+e^-$ Annihilation” (Sec. 36 of this Review).

References: