9. Quantum Chromodynamics

9.1. The QCD Lagrangian

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Quantum Chromodynamics (QCD), the gauge field theory which describes the strong interactions of colored quarks and gluons, is one of the components of the SU(3)×SU(2)×U(1) Standard Model. A quark of specific flavor (such as a charm quark) comes in 3 colors; gluons come in eight colors; hadrons are color-singlet combinations of quarks, anti-quarks, and gluons. The Lagrangian describing the interactions of quarks and gluons is (up to gauge-fixing terms)

\[ L_{\text{QCD}} = -\frac{1}{4} \epsilon^{\alpha\beta\gamma\delta} F_{\mu\nu}^{\alpha} F_{\mu\nu}^{\beta} \]

The QCD coupling constant, \( g \), and the gluons come in eight colors; hadrons are color-singlet combinations of quarks, anti-quarks, and gluons. The Lagrangian describing the interactions of quarks and gluons is (up to gauge-fixing terms)

\[ L_{\text{QCD}} = -\frac{1}{4} \epsilon^{\alpha\beta\gamma\delta} F_{\mu\nu}^{\alpha} F_{\mu\nu}^{\beta} \]

\[ \mu = 2\pi g_s^2 \]

\[ \beta_0 = 11 - \frac{2}{3} n_f \]

\[ \beta_1 = 19 - \frac{1}{9} n_f \]

\[ \beta_2 = 257 - \frac{5033}{9} n_f + \frac{325}{9} n_f^2 \]

where \( n_f \) is the number of quarks with mass less than the energy scale \( \mu \). The expression for the next term in this series \( \beta_3 \) can be found in Ref. 5. In solving this differential equation for \( \alpha_s \), a constant of integration is introduced. This constant is the one fundamental constant of QCD that must be determined from experiment. The most sensible choice for this constant is the value of \( \alpha_s \) at a fixed-reference scale \( \mu_0 \), which has become standard to choose \( \mu_0 = M_Z \). It is also convenient to introduce the dimensional parameter \( \Lambda \), since this provides a parameterization of the \( \mu \) dependence of \( \alpha_s \). The definition of \( \Lambda \) is arbitrary. One way to define it (adopted here) is to write a solution of Eq. (9.4) as an expansion in inverse powers of \( \ln(\mu^2) \):

\[ \alpha_s(\mu) = \frac{4\pi}{\beta_0} \ln(\mu^2/\Lambda^2) \left[ \frac{\beta_1}{\beta_0} \ln \left( \frac{\ln(\mu^2/\Lambda^2)}{\ln(\mu^2/\Lambda^2)} \right) + \frac{4\beta_2}{\beta_0^2} \ln^2(\mu^2/\Lambda^2) \right] \times \left( \ln \left( \frac{\ln(\mu^2/\Lambda^2)}{\ln(\mu^2/\Lambda^2)} \right) - \frac{1}{2} \beta_0 \right) \]

The last term in this expansion is

\[ \mathcal{O} \left( \frac{\ln^2(\mu^2/\Lambda^2)}{\ln(\mu^2/\Lambda^2)} \right) \]

and is usually neglected in the definition of \( \Lambda \). We choose to include it, For a fixed value of \( \alpha_s(M_Z) \), the inclusion of this term shifts the value of \( \Lambda \) by \( \sim 15 \text{ MeV} \). This solution illustrates the asymptotic freedom property: \( \alpha_s \to 0 \) as \( \mu \to \infty \).

Consider a “typical” QCD cross section which, when calculated perturbatively, starts at \( \mathcal{O}(\alpha_s) \):

\[ \sigma = A_1 \alpha_s + A_2 \alpha_s^2 + \cdots \]

The coefficients \( A_1, A_2 \) come from calculating the appropriate Feynman diagrams. In performing such calculations, various divergences arise, and these must be regulated in a consistent way. This requires a particular renormalization scheme (RS). The most commonly used one is the modified minimal subtraction (\( \overline{\text{MS}} \)) scheme [6]. This involves continuing momentum integrals from 4 to 4\(-2\) dimensions, and then subtracting off the resulting 1/\( \pi \) poles and also (\( \ln 4\pi - \gamma_E \)) which is another artifact of continuing the dimension. Here \( \gamma_E \) is the Euler-Mascheroni constant. To preserve the dimensionless nature of the coupling, a mass scale \( \mu \) must also be introduced: \( g \to \mu g \).

The finite coefficients \( A_1 \) (\( i \geq 2 \)) thus obtained depend implicitly on the renormalization convention used and explicitly on the scale \( \mu \).

The first two coefficients \( (\beta_0, \beta_1) \) in Eq. (9.4) are independent of the choice of RS. In contrast, the coefficients of terms proportional to \( \alpha_s^n \) for \( n > 3 \) are RS-dependent. The form given above for \( \beta_2 \) is in the \( \overline{\text{MS}} \) scheme.

The fundamental theorem of RS dependence is straightforward. Physical quantities, in particular the cross section, calculated to all orders in perturbation theory, do not depend on the RS. It follows that a truncated series does exhibit RS dependence. In practice, QCD cross sections are known to leading order (LO), or to next-to-leading order (NLO), or in a few cases, to next-to-next-to-leading order (NNLO); and it is only the latter two cases, which have reduced RS dependence, that are useful for precision tests. At NLO the RS dependence is completely given by one condition which can be taken to be the value of the renormalization scale \( \mu \). At NNLO this is not sufficient, and \( \mu \) is no longer equivalent to a choice of scheme; both must now be specified. One, therefore, has to address the question of what is the “best” choice for \( \mu \) within a given scheme, usually \( \overline{\text{MS}} \). There is no definite answer to this question—higher-order corrections do not “fix” the scale, rather they render the theoretical predictions less sensitive to its variation.

One should expect that choosing a scale \( \mu \) characteristic of the typical energy scale (\( E \)) in the process would be most appropriate. In general, a poor choice of scale generates terms of order \( \ln(E/\mu) \) in the \( A_i \)’s. Various methods have been proposed including choosing the scale for which the next-to-leading-order correction vanishes (“Fastest Apparent Convergence [7]”); the scale for which the next-to-next-to-leading order prediction is stationary [8], (i.e., the value of \( \mu \) where \( d\sigma/d\mu = 0 \)); or the scale dictated by the effective charge scheme [9] or by the BLM scheme [10]. By comparing the values of \( \alpha_s \) that different reasonable schemes give, an estimate of theoretical errors can be obtained. It has also been suggested to replace the perturbation series by its Pade approximant [11]. Results obtained using this method have, in certain cases, a reduced scale dependence [12,13]. One can also attempt to determine the scale from data by allowing it to vary and using a fit to determine it. This method can allow a determination of the error due to the scale choice and can give more confidence in the end result [14]. In many of the cases discussed below this scale uncertainty is the dominant error.

An important corollary is that if the higher-order corrections are naturally small, then the additional uncertainties introduced by the \( \mu \) dependence are likely to be small. There are some processes, however, for which the choice of scheme can influence the extracted value of \( \alpha_s(M_Z) \). There is no resolution to this problem other than to try to
calculate even more terms in the perturbation series. It is important to note that, since the perturbation series is an asymptotic expansion, there is a limit to the precision with which any theoretical quantity can be calculated. In some processes, the highest-order perturbative terms may be comparable in size to nonperturbative corrections (sometimes called higher-twist or renormalon effects, for a discussion see [15]); an estimate of these terms and their uncertainties is required if a value of $\alpha_s$ is to be extracted.

Cases occur where there is more than one large scale, say $\mu_1$ and $\mu_2$. In these cases, terms appear of the form $\log(\mu_1/\mu_2)$. If the ratio $\mu_1/\mu_2$ is large, these logarithms can render naive perturbation theory unreliable and a modified perturbation expansion that takes these terms into account must be used. A few examples are discussed below.

In the cases where the higher-order corrections to a process are known and are large, some caution should be exercised when quoting the value of $\alpha_s$. In what follows, we will attempt to indicate the size of the theoretical uncertainties on the extracted value of $\alpha_s$. There are two simple ways to determine this error. First, we can estimate it by comparing the value of $\alpha_s(\mu)$ obtained by fitting data using the QCD formula to highest known order in $\alpha_s$, and then comparing it with the value obtained using the next-to-highest-order formula ($\mu$ is chosen as the typical energy scale in the process). The corresponding $\Lambda_s$ are then obtained by evolving $\alpha_s(\mu)$ to $\mu = M_Z$ using Eq. (9.4) to the same order in $\alpha_s$ as the fit. Alternatively, we can vary the value of $\mu$ over a reasonable range, extracting a value of $\Lambda$ for each choice of $\mu$. This method is by its nature imprecise, since “reasonable” involves a subjective judgment. In either case, if the perturbation series is well behaved, the resulting error on $\alpha_s(M_Z)$ will be small.

In the above discussion we have ignored quark-mass effects, i.e., we have assumed an idealized situation where quarks of mass greater than $\mu$ are neglected completely. In this picture, the $\beta$-function coefficients change by discrete amounts as flavor thresholds (a quark of mass $M$) are crossed when integrating the differential equation for $\alpha_s$. Now imagine an experiment at energy scale $\mu$; for example, this could be $e^+e^- \rightarrow$ hadrons at center-of-mass energy $\mu$. If $\mu \gg M$, the mass $M$ is negligible and the process is well described by QCD with $n_f$ massless flavors and its parameter $\alpha_s(\mu)$ up to terms of order $\mu^2/M^2$. Conversely if $\mu \ll M$, the heavy quark plays no role and the process is well described by QCD with $n_f - 1$ massless flavors and its parameter $\alpha_s(n_f - 1)$ up to terms of order $\mu^2/M^2$. If $\mu \sim M$, the effects of the quark mass are process-dependent and cannot be absorbed into the running coupling. The values of $\alpha_s(n_f)$ and $\alpha_s(n_f - 1)$ are related so that a physical quantity calculated in both “theories” gives the same result [16]. This implies

$$\alpha_s(n_f)(M) = \alpha_s(n_f - 1)(M) - \frac{7}{12\pi^2} \alpha_s^2(n_f - 1)(M)$$  (9.7)

which is almost identical to the naive result $\alpha_s(n_f)(M) = \alpha_s(n_f - 1)(M)$. Here $M$ is the mass of the value of the running quark mass defined in the $\overline{MS}$ scheme (see the note on “Quark Masses” in the Particle Listings for more details), i.e., where $\Lambda(\alpha_s(M)) = M$.

It also follows that, for a relationship such as Eq. (9.5) to remain valid for all values of $\mu$, $\Lambda$ must also change as flavor thresholds are crossed, the value corresponds to an effective number of massless quarks: $\Lambda \rightarrow \Lambda(n_f)$ [16,17]. The formulae are given in the previous edition of this review.

An alternative matching procedure can be used [18]. This procedure requires the equality $\alpha_s(\mu)^{\alpha_f} = \alpha_s(\mu)^{\alpha_f - 1}$ for $\mu = M$. This matching is somewhat arbitrary; a different relation between $\Lambda^{\alpha_f}$ and $\Lambda^{\alpha_f - 1}$ would result if $\mu = M/2$ were used. In practice, the differences between these procedures are very small. $\Lambda^{(0)} = 200 \text{ MeV}$ corresponds to $\Lambda^{(1)} = 290 \text{ MeV}$ in the scheme of Ref. 18 and $\Lambda^{(1)} = 280 \text{ MeV}$ in the scheme we adopt. Note that the differences between $\Lambda^{(0)}$ and $\Lambda^{(1)}$ are numerically very significant.

Data from deep-inelastic scattering are in a range of energy where the bottom quark is not readily excited, and hence, these experiments quote $\Lambda(\alpha_s^{(1)})$. Most data from PEP, PETRA, TRISTAN, LEP, and SLC quote a value of $\Lambda(\alpha_s^{(1)})$ since these data are in an energy range where the bottom quark is light compared to the available energy. We have converted it to $\Lambda(\alpha_s^{(1)})$ as required. A few measurements, including the lattice gauge theory values from the $J/\psi$ system, and from $\gamma$ decay are at sufficiently low energy that $\Lambda(\alpha_s^{(1)})$ is appropriate.

In order to compare the values of $\alpha_s$ from various experiments, they must be evolved using the renormalization group to a common scale. For convenience, this is taken to be the mass of the $Z$ boson. This evolution uses third-order perturbation theory and can introduce additional errors particularly if extrapolation from very small scales is used. The variation in the charm and bottom quark masses ($m_c = 4.3 \pm 0.2 \text{ GeV}$ and $m_b = 1.3 \pm 0.3 \text{ GeV}$ are used) can also introduce errors. These result in a fixed value of $\alpha_s(2 \text{ GeV})$ giving an uncertainty in $\alpha_s(M_Z) = \pm 0.002$ if only perturbative evolution is used. There could be additional errors from nonperturbative effects that enter at low energy.

### 9.3. QCD in deep-inelastic scattering

The original and still one of the most powerful quantitative tests of perturbative QCD is the breaking of Bjorken scaling in deep-inelastic lepton-hadron scattering. In the leading-logarithm approximation, the measured structure functions $F_i(x, Q^2)$ are related to the quark distribution functions $q_i(x, Q^2)$ according to the naive parton model, by the formulae in “Cross-section Formulae for Specific Processes,” Sec. 35 of this Review. (In that section, $q_i$ is denoted by the notation $f_i$.) In describing the way in which scaling is broken in QCD, it is convenient to define nonsinglet and singlet quark distributions:

$$F_{NS} = q_1 - q_2 \quad F^S = \sum_i (q_i + \bar{q}_i).$$  (9.8)

The nonsinglet structure functions have nonzero values of flavor quantum numbers such as isospin or baryon number. The variation with $Q^2$ of these is described by the so-called DGLAP equations [19,20]:

$$Q^2 \frac{dF_{NS}}{dQ^2} = \frac{\alpha_s(Q)}{2 \pi} (p_{qq} \ast F_{NS})$$  (9.9a)

$$Q^2 \frac{dF^S}{dQ^2} = \frac{\alpha_s(Q)}{2 \pi} \left( \frac{p_{qq} + 2p_{gg}p_{qg}}{p_{qq}} \right) \ast F^S$$  (9.9b)

where $\ast$ denotes a convolution integral:

$$f \ast g = \int_x^y \frac{dy}{y} f(y) g \left( \frac{x}{y} \right).$$  (9.10)

The leading-order Altarelli-Parisi [20] splitting functions are

$$p_{qq} = \begin{cases} \frac{4}{3} \left( \frac{1 + x^2}{(1-x)^2} \right) + 2\delta(1-x), & (9.11a) \\ \frac{1}{2} \left[ x^2 + (1-x)^2 \right], & (9.11b) \\ \frac{3}{8} \left( \frac{1 + (1-x)^2}{x} \right), & (9.11c) \end{cases}$$

$$p_{gg} = \begin{cases} 0 & \left[ \frac{1-x}{x} + x(1-x) + \frac{x}{(1-x)} + \frac{11}{12}(1-x) \right] \\ -\frac{n_f}{3}\delta(1-x) & \end{cases}$$  (9.11d)

Here the gluon distribution $G(x, Q^2)$ has been introduced and $1/(1-x)\ast$ means

$$\int_0^1 \frac{dx}{(1-x)+} \ast \int_0^1 \frac{dx}{1-x} \ast f(x) = f(1).$$  (9.12)

The precision of contemporary experimental data demands that higher-order corrections also be included [21]. The above results are for massless quarks. At low $Q^2$ values, there are also important “higher-twist” (HT) contributions of the form:

$$F_i(x, Q^2) = F_i^{(LT)}(x, Q^2) + \frac{F_i^{(HT)}(x, Q^2)}{Q^2} + \cdots.$$  (9.13)
Leading twist (LT) indicates a term whose behavior is predicted by perturbative QCD. These corrections are numerically important only for $Q^2 < \mathcal{O}(\text{few GeV}^2)$ except for $x$ very close to 1. At very large values of $x$ perturbative corrections proportional to $\log(1-x)$ can become important [22].

A detailed review of the current status of the experimental data can be found, for example, in Refs. [23-26], and only a brief summary will be presented here. We shall only include determinations of $\alpha_s$ from the recently published results; the earlier editions of this Review should be consulted for the earlier data. Data now exist from HERA at much smaller values of $x$ than the fixed-target data. They provide valuable information about the shape of the antiquark and gluon distribution functions at $x \sim 10^{-4}$ [27].

From Eq. (9.9), it is clear that a nonsinglet structure function offers in principle the most precise test of the theory, since the $Q^2$ evolution is independent of the unmeasured gluon distribution. The CCFR collaboration fit the Gross-Llewellyn Smith sum rule [28] to the NLO terms which are required to fix the $Q^2$ scale. The data are dominated by 4 GeV$^2 < Q^2 < 100$ GeV$^2$. The fit [55] using H1 data [56] gives $\alpha_s(M_Z) = 0.122 \pm 0.004$ (exp.) $\pm 0.009$ (theory). (The theoretical error is taken from Ref. 54.) The dominant part of the theoretical error is from the scale dependence; errors from terms that are suppressed by $1/\log(1/x)$ in the quark sector are included [57] while those from the gluon sector are not.

Typically, $\Lambda$ is extracted from the deep inelastic scattering data by parameterizing the parton densities in a simple analytic way at some $Q_0^2$, evolving to higher $Q^2$ using the next-to-leading-order evolution equations, and fitting globally to the measured structure function data to obtain $\Delta \alpha_s(Q_0^2)$. Thus, an important by-product of such studies is the extraction of parton densities at a fixed-reference value of $Q_0^2$. These can then be evolved in $Q^2$ and used as input for phenomenological studies in hadron-hadron collisions (see below).

To avoid having to evolve from the starting $Q_0^2$ value each time, a parton density is required; it is useful to have available a simple analytic approximation to the densities valid over a range of $x$ and $Q^2$ values. A package is available from the CERN computer library that includes an exhaustive set of fits [58]. Most of these fits are obsolete.

In using a parameterization to predict event rates, a next-to-leading order fit must be used if the process being calculated is known to next-to-leading order in QCD perturbation theory. In such a case, there is an additional scheme dependence; this scheme dependence is reflected in the $\mathcal{O}(\alpha_s)$ corrections that appear in the relations between the structure functions and the quark distribution functions. There are two common schemes: a deep-inelastic scheme where there are no order $\alpha_s$ corrections in the formula for $F_2(x,Q^2)$ and the minimal subtraction scheme. It is important when these next-to-leading order fits are used in other processes (see below), that the same scheme is used in the calculation of the partonic rates. Most current sets of parton distributions are obtained using fits to all relevant event data [59]. In particular, data from purely hadronic initial states are used as they can provide important constraints on the gluon distributions.

9.4. QCD in decays of the $\tau$ lepton

The semi-leptonic branching ratio of the tau ($\tau \to \nu_\tau + \text{hadrons}$, $R_\tau$) is an inclusive quantity. It is related to the contribution of hadrons to the imaginary part of the $W$ self energy ($\Pi(s)$). It is sensitive to a range of energies since it involves an integral

$$R_\tau \sim \int_0^{m_\tau^2} \frac{d s}{m_\tau^2} \left( 1 - \frac{s}{m_\tau^2} \right) \text{Im} \left( \Pi(s) \right).$$

Since the scale involved is low, one must take into account nonperturbative (higher-twist) contributions which are suppressed by powers of the $\tau$ mass.

$$R_\tau \approx 3.058 \left[ 1 + \frac{\alpha_s(m_\tau)}{\pi} + 5.2 \left( \frac{\alpha_s(m_\tau)}{\pi} \right)^2 + 26.4 \left( \frac{\alpha_s(m_\tau)}{\pi} \right)^3 \right. \left. + \frac{a}{m_\tau^2} + b \frac{\ln M_t}{m_\tau^2} + c \frac{\ln M_t}{m_\tau^2} + \cdots \right].$$

(9.15)

Here $a$, $b$, and $c$ are dimensionless constants and $m$ is a light quark mass. The term of order $1/m_\tau^2$ is a kinematical effect due to the light quark masses and is consequently very small. The nonperturbative terms are estimated using sum rules [60]. In total, they are estimated to be $\sim 0.014 \pm 0.005$ [61,62]. This estimate relies on there being no
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Tevatron collider data are fitted over a very large range of transverse momenta, to the QCD prediction for the underlying scattering process that depends on $\alpha_s$. The evolution of the coupling over this energy range (40 to 250 GeV) is therefore tested in the analysis. CDF obtains $\alpha_s(M_Z) = 0.1129 \pm 0.0001$ (stat.) $\pm 0.0085$ (syst.) [74]. Estimation of the theoretical errors is not straightforward. The structure functions depend implicitly on $\alpha_s$ and an iteration procedure must be used to obtain a consistent result; different sets of structure functions yield different correlations between the two values of $\alpha_s$. I estimate an uncertainty of $\pm 0.005$ from examining the fits. Ref. 73 estimates the error from unknown higher order QCD corrections to be $\pm 0.005$. Combining these then gives, $\alpha_s(M_Z) = 0.113 \pm 0.011$. Data are also available on the angular distribution of jets; these are also in agreement with QCD expectations [75,76].

QCD corrections to Drell-Yan type cross sections (i.e., the production in hadron collisions by quark-antiquark annihilation of lepton pairs of invariant mass $Q$ from virtual photons, or of real $W$ or $Z$ bosons), are known [77]. These $\mathcal{O}(\alpha_s)$ QCD corrections are sizable at small values of $Q$. The correction to $W$ and $Z$ production, as measured in $\gamma g$ collisions at $\sqrt{s} = 63$ TeV and $\sqrt{s} = 1.8$ TeV, is of order 30%. The NNLO corrections to this process are known [78].

The production of $W$ and $Z$ bosons and photons at large transverse momentum can also be used to test QCD. The leading-order QCD subprocesses are $\gamma g \to \gamma g$ and $qg \to \gamma q$. If the parton distributions are taken from other processes and a value of $\alpha_s$ assumed, then an absolute prediction is obtained. Conversely, the data can be used to extract information on quark and gluon distributions and on the value of $\alpha_s$. The next-to-leading-order QCD corrections are known [79,80] (for photons), and for $W/Z$ production [81], and so a precision test is possible. Data exist on photon production from the CDF and D0 collaborations [82,83] and from fixed target experiments [84]. Detailed comparisons with QCD predictions [85] may indicate an excess of the data over the theoretical prediction at low value of transverse momenta, although other authors [86] find smaller excesses.

The UA2 collaboration [87] extracted a value of $\alpha_s(M_W) = 0.123 \pm 0.018$ (stat.) $\pm 0.017$ (syst.) from the measured ratio $R_W = \sigma(W + 1 \text{jet}) / \sigma(W + 1 \text{jet})$. The result depends on the algorithm used to define a jet, and the dominant systematic errors due to fragmentation and corrections for underlying events (the former causes jet energy to be lost, the latter causes it to be increased) are connected to the algorithm. There is also dependence on the parton distribution functions, and hence, $\alpha_s$ appears explicitly in the formula for $R_W$, and implicitly in the distribution functions. This result is not used in the final average. Data from CDF and D0 on the $W$ $p_T$ distribution [89] are in agreement with QCD but are not able to determine $\alpha_s$ with sufficient precision to have any weight in a global average.

In the region of low $p_T$, fixed order perturbation theory is not applicable; one must sum terms of order $\alpha_s^n \ln^n(p_T/M_H)$ [88]. Data from D0 [90] on the $p_T$ distribution of $Z$ bosons agree well with these predictions.

The production rates of $b$ quarks in $\gamma g$ have been used to determine $\alpha_s$ [91]. The next-to-leading-order QCD production processes have been used. By selecting events where the $b$ quarks are back-to-back in azimuth, the next-to-leading-order calculation can be used to compare rates to the measured value and a value of $\alpha_s$ extracted. The errors are dominated by the measurement errors, the choice of $\mu$ and $M$, and uncertainties in the choice of structure functions. The last were estimated by varying the structure functions used. The result is $\alpha_s(M_Z) = 0.113^{+0.010}_{-0.009}$. The UA1 collaboration [87] extracted a value of $\alpha_s(M_W) = 0.123 \pm 0.018$ (stat.) $\pm 0.017$ (syst.) from the measured ratio $R_W = \sigma(W + 1 \text{jet}) / \sigma(W + 1 \text{jet})$. The result depends on the algorithm used to define a jet, and the dominant systematic errors due to fragmentation and corrections for underlying events (the former causes jet energy to be lost, the latter causes it to be increased) are connected to the algorithm. There is also dependence on the parton distribution functions, and hence, $\alpha_s$ appears explicitly in the formula for $R_W$, and implicitly in the distribution functions. This result is not used in the final average. Data from CDF and D0 on the $W$ $p_T$ distribution [89] are in agreement with QCD but are not able to determine $\alpha_s$ with sufficient precision to have any weight in a global average.

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9.6. QCD in heavy-quarkonium decay

Under the assumption that the hadronic and leptonic decay widths of heavy QCD resonances can be factorized into a nonperturbative part—dependent on the confining potential—and a calculable perturbative part, the ratios of partial decay widths allow measurements of $\alpha_s$ at the heavy-quark mass scale. The most precise data come from the decay widths of the $1^−−$ J/$\psi(1S)$ and $\Upsilon$ resonances. The total decay width of the $\Upsilon$ is predicted by perturbative QCD [93]

$$R_\Upsilon(T) = \frac{\Gamma(\Upsilon \to \text{hadrons})}{\Gamma(\Upsilon \to \mu^+\mu^-)} = \frac{10(\alpha^2 - 9)\alpha_s^2(M)}{9\pi\alpha_s^2} \times \left[ 1 + \frac{3\beta_0}{2} \left( 1.162 + \ln\left( \frac{2M}{M_T} \right) \right) \right].$$

(9.16)

Data are available for the $\Upsilon$, $\Upsilon'$, $\Upsilon''$, and J/$\psi$. The result is very sensitive to $\alpha_s$ and the data are sufficiently precise ($R_\Upsilon(T) = 32.5 \pm 0.9$) [94] that the theoretical errors will dominate. There are theoretical corrections to this simple formula due to the relativistic nature of the $\Upsilon$ system; $\alpha^2/\alpha^2 \sim 0.1$ for the $\Upsilon$. They are more severe for the J/$\psi$. There are also nonperturbative corrections of the form $A^2/M^2$: again these are more severe for the J/$\psi$. A fit to $\Upsilon$, $\Upsilon'$, and $\Upsilon''$ [95] gives $\alpha_s(M_Z) = 0.113 \pm 0.001$ (expt.) and from scale separately and also from the $J/\psi$ are consistent with each other. There is an uncertainty of order $\pm 0.005$ from the choice of scale; the error from $\alpha^2/\alpha^2$ corrections is a little larger. The ratio

$$\frac{T_{\Upsilon} - T_{\text{hadrons}}}{T_{\Upsilon} - T_{\text{hadrons}}}$$

has been measured by the CLEO collaboration who use it to determine $\alpha_s(9.45 \text{ GeV}) = 0.163 \pm 0.002 \pm 0.014$ [97] which corresponds to $\alpha_s(M_Z) = 0.110 \pm 0.001 \pm 0.007$. The error is dominated by theoretical uncertainties associated with the scale choice. The theoretical uncertainties due to the production of photons in fragmentation [96] are small [97]. Higher order QCD calculations of the photon energy distribution are available [98]; this distribution could now be used to further test the theory. The width $\Gamma(T \to e^+e^-)$ can also be used to determine $\alpha_s$ by using moments of the quantity $R_\Upsilon(s) = \frac{\sigma(e^+e^- \to bb)}{\sigma(e^+e^- \to \mu^+\mu^-)}$ defined by $M_n = \int_0^{M_n} R_\Upsilon(s) ds$. At large values of $n$, $M_n$ is dominated by $\Gamma(T \to e^+e^-)$. Higher order corrections are available and the method [100] $\alpha_s(M_t) = 0.220 \pm 0.027$. The dominant error is theoretical and is dominated by the choice of scale and by uncertainties in Coulomb corrections. It corresponds to $\alpha_s(M_Z) = 0.119 \pm 0.008$. These various $T$ measurements can be combined and give $\alpha_s(M_Z) = 0.114 \pm 0.008$.

9.7. Perturbative QCD in $e^+e^-$ collisions

The total cross section for $e^+e^- \to \text{hadrons}$ is obtained (at low values of $\sqrt{s}$) by multiplying the muon-pair cross section by the factor $R = 2\Sigma_q^2$. The higher-order QCD corrections to this quantity have been calculated, and the results can be expressed in terms of the factor:

$$R = R^{(0)} \left[ 1 + \frac{\alpha_s}{\pi} + C_2 \left( \frac{\alpha_s}{\pi} \right)^2 + C_3 \left( \frac{\alpha_s}{\pi} \right)^3 + \cdots \right],$$

(9.17)

where $C_2 = 1.411$ and $C_3 = -12.8$ [101].

$R^{(0)}$ can be obtained from the formula for $d^2\sigma/d\Omega ds$ for $e^+e^- \to f\bar{f}$ by integrating over $\Omega$. The formula is given in Sec. 35.2 of this Review. This result is only correct in the zero-quark-mass limit. The $O(\alpha_s)$ corrections are also known for massive quarks [102]. The principal advantage of determining $\alpha_s$ from $R$ is that the $e^+e^-$ annihilation is that there is no dependence on fragmentation models, jet algorithms, etc.

A measurement by CLEO [103] at $\sqrt{s} = 10.52 \text{ GeV}$ yields $\alpha_s(10.52 \text{ GeV}) = 0.20 \pm 0.01 \pm 0.06$, which corresponds to $\alpha_s(M_Z) = 0.13 \pm 0.005 \pm 0.03$. A comparison of the theoretical prediction of Eq. (9.17) (corrected for the $b$-quark mass), with all the available data at values of $\sqrt{s}$ between 20 and 65 GeV, gives [104] $\alpha_s(35 \text{ GeV}) = 0.146 \pm 0.030$. The size of the order $\alpha_s^3$ term is of order 40% of that of the order $\alpha_s^2$ and 3% of the order $\alpha_s$. If the order $\alpha_s^2$ term is not included, a fit to the data yields $\alpha_s(34 \text{ GeV}) = 0.142 \pm 0.03$, indicating that the theoretical uncertainty is smaller than the experimental error.

Measurements of the ratio of hadronic to leptonic width of the $Z$ at LEP and SLC, $\Gamma_h/\Gamma_{1\mu}$, probe, the same quantity as $R$. Using the average of $\Gamma_h/\Gamma_{1\mu} = 20.783 \pm 0.029$ gives $\alpha_s(M_Z) = 0.123 \pm 0.004$ [105]. There are theoretical errors arising from the values of top-quark and Higgs masses which enter due to electroweak corrections to the $Z$ width and from the choice of scale.

While this method has small theoretical uncertainties from QCD itself, it relies sensitively on the electroweak couplings of the $Z$ to quarks [106]. The presence of new physics which changes these couplings via electroweak radiative corrections would invalidate the value of $\alpha_s(M_Z)$. However, given the excellent agreement [107] of the many measurements at the $Z$, there is no reason not to use the value of $\alpha_s(M_Z) = 0.1192 \pm 0.0028 \pm 0.002(\text{scale})$ from the global fits of the various precision measurements at LEP/SLC and the W and top masses in the world average (see the section on “Electroweak model and constraints on new physics.” Sec. 10 of this Review).

An alternative method of determining $\alpha_s$ in $e^+e^-$ annihilation is from measuring quantities that are sensitive to the relative rates of two-, three-, and four-jet events. A review should be consulted for more details [108] of the issues mentioned briefly here. In addition to simply counting jets, there are many possible choices of such “shape variables”: thrust [109], energy-energy correlations [110], average jet mass, etc. All of these are infrared safe, which means they can be reliably calculated in perturbation theory. The starting point for all these quantities is the multijet cross section. For example, at order $\alpha_s$, for the process $e^+e^- \to qqqq$ [111]

$$\frac{1}{\sigma} \frac{d^2\sigma}{dz_1 dz_2} = \frac{2\alpha_s}{3\pi} x_1^2 + x_2^2 \left( 1 - x_1 \right) \left( 1 - x_2 \right)$$

(9.18)

$$x_i = \frac{2E_i}{\sqrt{s}}$$

(9.19)

where $x_i$ are the center-of-mass energy fractions of the final-state (massless) quarks. A distribution in a “three-jet” variable, such as those listed above, is obtained by integrating this differential cross section over an appropriate phase space region for a fixed value of the variable. The order $\alpha_s^2$ corrections to this process have been computed, as well as the 4-jet final states such as $e^+e^- \to qqqq$ [112].

There are many methods used by the $e^+e^-$ experimental groups to determine $\alpha_s$ from the event topology. The jet-counting algorithm, originally introduced by the JADE collaboration [113], has been used by many other groups. Here, particles of momenta $p_1$ and $p_2$ are combined into a pseudo-particle of momentum $p_1 + p_2$ if the invariant mass of the pair is less than $0.07\sqrt{s}$. The process is then iterated until no more pairs of particles or pseudo-particles remain. The remaining number is then defined to be the number of jets in the event, and can be compared to the QCD prediction. The Durham algorithm is slightly different: in computing the mass of a pair of partons, it uses $M^2 = 2\min(E_i^2, E_j^2)(1 - \cos\theta_{ij})$ for partons of energies $E_i$ and $E_j$ separated by angle $\theta_{ij}$ [114].

There are theoretical ambiguities in the way this process is carried out. Quarks and gluons are massless, whereas the observed hadrons are not, so that the massive jets that result from this scheme cannot be cycled directly to the massless jets of perturbative QCD. Different recombination schemes have been tried, for example combining 3-momenta and then rescaling the energy of the cluster so that it remains massless. These schemes result in the same data giving a slightly different values [115,116] of $\alpha_s$. These differences can be used to determine a systematic error. In addition, since what is observed are hadrons rather than quarks and gluons, a model is needed to describe the evolution of a partonic final state into one involving hadrons, so that detector corrections can be applied. The QCD matrix elements are combined with a parton-fragmentation
model. This model can then be used to correct the data for a direct comparison with the parton calculation. The different hadronization models that are used [117–120] model the dynamics which are controlled by nonperturbative QCD effects which we cannot yet calculate. The fragmentation parameters of these Monte Carlos are tuned to get agreement with the observed data. The differences between these models contribute to the systematic errors. The systematic errors from recombination schemes and fragmentation effects dominate over the statistical and other errors of the LEP/SLD experiments.

The scale $M$ at which $\alpha_s(M)$ is to be evaluated is not clear. The invariant mass of a typical jet (or $\sqrt{p_t}$) is probably a more appropriate choice than the $e^+e^-$ center-of-mass energy. While there is no justification for doing so, if the value is allowed to float in the fit to the data, the fit improves and the data tend to prefer values of order $\sqrt{10}$ GeV for some variables [116,121]; the exact value depends on the variable that is fitted.

The perturbative QCD formulae can break down in special kinematical configurations. For example, the thrust ($T$) distribution contains terms of the type $\alpha_s^n$ln$(1−T)$. The higher orders in the perturbation expansion contain terms of order $\alpha_s^n$ln$m$$(1−T)$. For $T \sim 1$ (the region populated by 2-jet events), the perturbation expansion is unreliable. The terms with $n \leq m$ can be summed to all orders in $\alpha_s$ [122]. If the jet recombination methods are used higher-order terms involve $\alpha_s^n$ln$(m)/(m)$; these too can be resummed [123]. The resummed results give better agreement with the data at large values of $T$. Some caution should be exercised in using these resummed results because of the possibility of overcounting; the showering Monte Carlos that are used for the fragmentation corrections also generate some of these leading-log corrections. Different schemes for combining the order $\alpha_s^n$ and the resumptions are available [124]. These different schemes result in shifts in $\alpha_s$ at order $\alpha_s^n$ and the resummations are available [124]. The use of the resummed results improves the agreement between the data and the theory. An average of the recent results at the $Z$ resonance from SLD [116], OPAL [125], L3 [126], ALEPH [127], and DELPHI [128], using the combined $\alpha_s^n$ and resummation fitting to a large set of shape variables, gives $\alpha_s(M_Z)=0.122\pm0.007$. The errors in the values of $\alpha_s(M_Z)$ from these shape variables are totally dominated by the theoretical uncertainties associated with the choice of scale, and the effects of hadronization Monte Carlos on the different quantities fitted.

Similar studies on event shapes have been undertaken at lower energies at TRISTAN, PEP/PETRA, and CLEO. A combined result of various shape parameters by the TOPAZ collaboration gives $\alpha_s(58 \text{ GeV})=0.125\pm0.009$, using the fixed order QCD result, and $\alpha_s(58 \text{ GeV})=0.132\pm0.008$ (corresponding to $\alpha_s(M_Z)=0.123\pm0.007$, using the same method as in the SLD and LEP average [129]. The measurements of event shapes at PEP/PETRA are summarized in earlier editions of this note. A recent reevaluation of the JADE data [130] obtained using resummed QCD results and by averaging over several shape variables gives $\alpha_s(35 \text{ GeV})=0.149^{+0.012}_{−0.007}$. An analysis by the TPC group [131] gives $\alpha_s(29 \text{ GeV})=0.160\pm0.012$, using the same method as TOPAZ. This value corresponds to $\alpha_s(M_Z)=0.131\pm0.010$.

The CLEO collaboration fits to the order $\alpha_s^n$ results for the two jet fraction at $\sqrt{s}=10.53 \text{ GeV}$, and obtains $\alpha_s(10.53 \text{ GeV})=0.164\pm0.004$ (expt. $\pm0.014$ (theory) [132]. The dominant systematic error arises from the choice of scale ($\mu$), and is determined from the range of $\alpha_s$ that results from fit with $\mu=10.53 \text{ GeV}$, and a fit where $\mu$ is allowed to vary to get the lowest $\chi^2$. The latter results in $\mu=1.2\text{ GeV}$. Since the quoted result corresponds to $\alpha_s(1.2\text{ GeV})=0.35$, it is by no means clear that the perturbative QCD expression is reliable and the resulting error should, therefore, be treated with caution. A fit to many different variables as is done in the LEP/SLC analyses would give added confidence to the quoted error.

Recently studies have been carried out at energies between $\sim130 \text{ GeV}$ [133] and $\sim189 \text{ GeV}$ [134]. These can be combined to give $\alpha_s(130 \text{ GeV})=0.114\pm0.008$ and $\alpha_s(189 \text{ GeV})=0.1104\pm0.005$. The dominant errors are theoretical and systematic and, as most of these are in common at the two energies. These data and those at the $Z$ resonance provide clear confirmation of the expected decrease in $\alpha_s$ as the energy is increased.

Since the errors in the event shape measurements are dominantly systematic, and are common to the experiments, the results from PEP/PETRA, TRISTAN, LEP, SLC, and CLEO are combined to give $\alpha_s(M_Z)=0.121\pm0.007$. All of the experiments are consistent with this average and, taken together, provide verification of the running of the coupling constant with energy.

Estimates are available for the nonperturbative corrections to the mean value of $1−T$ [136]. These are of order $1/E$ and involve a single parameter to be determined from experiment. These corrections can then be used as an alternative to those modeled by the fragmentation Monte Carlos. The DELPHI collaboration [135] uses data up to the $Z$ mass from many experiments and determines $\alpha_s(M_Z)=0.119\pm0.006$, the error being dominated by the choice of scale. The values is also determined by a fit to a second variable (the mean jet mass); while the extracted values of $\alpha_s(M_Z)$ are consistent with each other, the values of the non perturbative parameter are not. The analysis is useful as one can directly determine the size of the $1/E$ corrections; they are approximately 20% (50%) of the perturbative result at $\sqrt{s}=9(11)$ GeV.

9.8. Scaling violations in fragmentation functions

Measurements of the fragmentation function $d_F(z,E)$, (the probability that a hadron of type $i$ be produced with energy $E$ in $e^+e^-$ collisions at $\sqrt{s}=2E$) can be used to determine $\alpha_s$. As in the case of scaling violations in structure functions, QCD predicts only the $E$ dependence. Hence, measurements at different energies are needed to extract a value of $\alpha_s$. Because the QCD evolution mixes the fragmentation functions for each quark flavor with the gluon fragmentation function, it is necessary to determine each of these before $\alpha_s$ can be extracted. The ALEPH collaboration has used data from energies ranging from $\sqrt{s}=22 \text{ GeV}$ to $\sqrt{s}=91 \text{ GeV}$. A flavor tag is used to discriminate between different quark species, and the longitudinal and transverse cross sections are used to extract the gluon fragmentation function [137]. The result obtained is $\alpha_s(M_Z)=0.126\pm0.007$ (expt.) $\pm0.006$ (theory) [138]. The theory error is due mainly to the choice of scale. The OPAL collaboration [139] has also extracted the separate fragmentation functions. DELPHI [140] has also performed a similar analysis using data from other experiments at lower energy with the result $\alpha_s(M_Z)=0.124\pm0.007\pm0.009$ (theory). The larger theoretical error is due to the larger range of scales that were used in the fit. These results can be combined to give $\alpha_s(M_Z)=0.125\pm0.005\pm0.008$ (theory).

9.9. Photon structure functions

$e^+e^-$ can also be used to study photon-photon interactions, which can be used to measure the structure function of a photon [141], by selecting events of the type $e^+e^−\to e^+e^-+\text{hadrons}$ which proceeds via two photon scattering. If events are selected where one of the photons is almost on mass shell and the other has a large invariant mass $Q$, then the latter probes the photon structure function at scale $Q$; the process is analogous to deep inelastic scattering where a highly virtual photon is used to probe the proton structure. This process was included in earlier versions of this Review which can be consulted for details on older measurements [142-145]. A recent review of the data can be found in Ref. 146. Data have become available from LEP [147-150] and from TRISTAN [151,152] which extend the range of $Q^2$ to of order 300 GeV$^2$ and $x$ as low as $2 \times 10^{-3}$ and show $Q^2$ dependence of the structure function that is consistent with QCD expectations. Experiments at HERA can also probe the photon structure function by looking at jet production in $\gamma p$ collisions; this is analogous to the jet production in hadron-hadron collisions which is sensitive to hadron structure functions. The data [153] are consistent with theoretical models [154].
9.10. Jet rates in $ep$ collisions

At lowest order in $\alpha_s$, the $ep$ scattering process produces a final state of $(1+1)$ jets, one from the proton fragment and the other from the quark knocked out by the process $e + \text{quark} \rightarrow e + \text{quark}$. At next order in $\alpha_s$, a gluon can be radiated, and hence a $(2+1)$ jet final state produced. By comparing the rates for these $(1+1)$ and $(2+1)$ jet processes, a value of $\alpha_s$ can be obtained. A NLO QCD calculation is available [155]. The basic methodology is similar to that used in the jet counting experiments in $e^+e^-$ annihilation discussed above. Unlike those measurements, the ones in ep scattering are not at a fixed value of $Q^2$. In addition to the systematic errors associated with the jet definitions, there are additional ones since the structure functions enter into the rate calculations. Results from H1 [156] and ZEUS [157] can be combined to give $\alpha_s(M_Z) = 0.118 \pm 0.005$ (stat.) $\pm 0.009$ (syst.). The contributions to the systematic errors from experimental effects (mainly the hadronic energy scale) are comparable to the theoretical ones arising from scale choice, structure functions, and jet definitions. The theoretical errors are common to the two measurements; therefore, we have not reduced the systematic error after forming the average.

9.11. QCD in diffractive events

In approximately 10% of the deep-inelastic scattering events at HERA a rapidity gap is observed [158]; that is events are seen where there are almost no hadrons produced in the direction of the incident proton. This was unexpected; QCD based models of the final state predicted that the rapidity interval between the quark that is hit by the electron and the proton remnant should be populated approximately evenly by the hadrons. Similar phenomena have been observed at the Tevatron in W and jet production. For a review see Ref. 159.

9.12. Lattice QCD

Lattice gauge theory calculations can be used to calculate, using non-perturbative methods, a physical quantity that can be measured experimentally. The value of this quantity can then be used to determine the QCD coupling that enters in the calculation. For a review of the methodology see Ref. 160. For example, the energy levels of a $Q\bar{Q}$ system can be determined and then used to extract $\alpha_s$. The masses of the $Q\bar{Q}$ states depend only on the quark mass and on $\alpha_s$. A limitation is that calculations cannot be performed for three light quark flavors. Results are available for zero ($n_f = 0$, quenched approximation) and two light flavors, which allow extrapolation to three. The coupling constant so extracted can be combined to give $\alpha_s$ using the strength of the force between two heavy quarks computed using Wilson fermions. These authors point out that their result is somewhat. Precision at the 1% level may be achievable if the systematic and theoretical errors can be reduced [170].

The result of averaging [163,161,164] gives with a more conservative error $\alpha_s(M_Z) = 0.115 \pm 0.003$. This will be used in the average.

9.13. Conclusions

The need for brevity has meant that many other important topics in QCD phenomenology have had to be omitted from this review. One should mention in particular the study of exclusive processes (form factors, elastic scattering, ...), the behavior of quarks and gluons in nuclei, the spin properties of the theory, and QCD effects in hadron spectroscopy.

We have focused on those high-energy processes which currently offer the most quantitative tests of perturbative QCD. Figure 9.1 shows the values of $\alpha_s(M_Z)$ deduced from the various experiments. Figure 9.2 shows the values and the error on the average, assuming that all of the errors in the contributing results are uncorrelated, is $\pm 0.0014$, and may be an underestimate. Almost all of the values used in the average are dominated by systematic, usually theoretical, errors. Only some of these, notably from the choice of scale, are correlated. The average is not dominated by a single measurement; there are several results with comparable small errors: these are the ones from $\tau$ decay, lattice gauge theory, deep inelastic scattering, and the Z width. We quote our average value as $\alpha_s(M_Z) = 0.1181 \pm 0.002$, which corresponds to $\Lambda_s^{(5)} = 200^{+25}_{-20}$ MeV using Eq. (9.5e). Future experiments can be expected to improve the measurements of $\alpha_s$ somewhat. Precision at the 1% level may be achievable if the systematic and theoretical errors can be reduced [170].

The value of $\alpha_s$ at any scale corresponding to our average can be obtained from http://www-theory.lbl.gov/~ianh/alpha/alpha.html

Figure 9.2: Summary of the values of $\alpha_s(\mu)$ at the values of $\mu$ where they are measured. The lines show the central values and the $\pm 1\sigma$ limits of our average. The figure clearly shows the decrease in $\alpha_s(\mu)$ with increasing $\mu$. The data are, in increasing order of $\mu$, $\tau$ width, deep inelastic scattering, $\tau$ decays, $e^+e^-$ event rate at 25 GeV, event shapes at TRISTAN, $Z$ width, $e^+e^-$ event shapes of $M_Z$, 135, and 189 GeV.
References:

S. Catani, B.R. Webber, and G. Turnock, Phys. Lett. B272, 368 (1991);
128. P. Abreu et al., Z. Phys. C59, 21 (1993);
130. P.A. Movilla Fernandez et al., Eur. Phys. J. C1, 461 (1998);
131. D.A. Bauer et al., SLAC-PUB-6518.
134. ALEPH Collaboration: 99-032 (1999); DELPHI Collaboration: 99-114 (1999); L3 Collaboration: L3-2414 (1999); OPAL Collaboration, PN-403 (1999); all submitted to International Conference on Lepton Photon Interactions, Stanford, USA (Aug. 1999);
B. W. Harris and J. F. Owens, Phys. Rev. D56, 4007 (1997);
S. Catani and M. Seymour, Nucl. Phys. B485, 291 (1997);
A.X. El-Khadra et al., FNAL 94-091/T (1994);
165. S. Collins et al., cited by [166].
167. G.S. Bali and K. Schilling, Phys. Rev. D47, 661 (1973);