7. ELECTROMAGNETIC RELATIONS

Revised September 2005 by H.G. Spieler (LBNL).

<table>
<thead>
<tr>
<th>Conversion factors:</th>
<th>Gaussian CGS</th>
<th>SI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Charge:</td>
<td>2.997 921 58 × 10⁹ esu</td>
<td>= 1 C = 1 A s</td>
</tr>
<tr>
<td>Potential:</td>
<td>(1/299.792 458) statvolt (ergs/emu)</td>
<td>= 1 V = 1 J C⁻¹</td>
</tr>
<tr>
<td>Magnetic field:</td>
<td>10⁴ gauss = 10⁴ dyne/emu</td>
<td>= 1 T = 1 N A⁻¹ m⁻¹</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Linear media:</th>
<th>D = ϵE, H = B / µ</th>
<th>D = ϵ₀E + P, H = B / µ₀ - M</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>µ₀ = 4π × 10⁻⁷ N A⁻²</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>ϵ₀ = 8.854 187... × 10⁻¹² F m⁻¹</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Constitutive relations:</th>
<th>F = q(E + v x B)</th>
<th>F = q(E + v x B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>D = 4πρ</td>
<td>4πρ</td>
<td>ρ = 4πρ</td>
</tr>
<tr>
<td>∇ x H - (c/∂t) D</td>
<td>4πJ</td>
<td>J = c/∂t J</td>
</tr>
<tr>
<td>∇ x B = 0</td>
<td>∇ x E + (c/∂t) B = 0</td>
<td></td>
</tr>
<tr>
<td>∇ x E + (c/∂t) B = 0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Linear media:</th>
<th>E = -∇V - (c/∂t) A</th>
<th>E = -∇V - (c/∂t) A</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>B = ∇ x A</td>
<td>B = ∇ x A</td>
</tr>
</tbody>
</table>

| V = ∑ qᵢ/|rᵢ| | = 1/4πϵ₀ ∫ ρ(r')/|r - r'| d³x' | V = 1/4πϵ₀ ∫ ρ(r')/|r - r'| d³x' |
| A = 1/c ∫ J(r')/|r-r'| | = 1/4π ∫ J(r')/|r-r'| d³x' | A = 1/4π ∫ J(r')/|r-r'| d³x' |

| Eᵢ⁄ = Eᵢ⁄ | Eᵢ⁄ = Eᵢ⁄ |
| Eᵢ⁄ = γ(Eᵢ⁄ + c v x B) | Eᵢ⁄ = γ(Eᵢ⁄ + v x B) |
| Bᵢ⁄ = Bᵢ⁄ | Bᵢ⁄ = Bᵢ⁄ |
| Bᵢ⁄ = γ(Bᵢ⁄ - 1/c v x E) | Bᵢ⁄ = γ(Bᵢ⁄ - v x E) |

\[ -\frac{1}{4πϵ₀} = \frac{e²}{2.997 924 58 × 10⁸} \text{ m}^{-1} \text{ s}^{-1} ; \frac{μ₀}{4π} = 10⁻⁷ \text{ N A}^{-2} ; c = \frac{1}{\sqrt{μ₀ϵ₀}} = 2.997 924 58 × 10⁸ \text{ m s}^{-1} \]
7.1. Impedances (SI units)

\( \rho = \text{resistivity at room temperature in } 10^{-8} \, \Omega \cdot \text{m} \):

\(~ 1.7 \text{ for Cu} \quad ~ 5.5 \text{ for W} \)

\(~ 2.4 \text{ for Au} \quad ~ 73 \text{ for SS 304} \)

\(~ 2.8 \text{ for Al} \quad ~ 100 \text{ for Nichrome} \)

(Al alloys may have double the Al value.)

For alternating currents, instantaneous current \( I \), voltage \( V \), angular frequency \( \omega \):

\[ V = V_0 \, e^{j \omega t} = Z I . \]  

(7.1)

Impedance of self-inductance \( L \):

\[ Z = j \omega L . \]

Impedance of capacitance \( C \):

\[ Z = 1/j \omega C . \]

Impedance of free space:

\[ Z = \sqrt{\mu_0/\varepsilon_0} = 376.7 \, \Omega . \]

High-frequency surface impedance of a good conductor:

\[ Z = (1 + j) \frac{\rho}{\delta} , \quad \text{where} \quad \delta = \text{skin depth} ; \]

\[ \delta = \sqrt{\frac{\rho}{\pi \mu_0}} \approx \frac{6.6}{\sqrt{\nu} \text{ (Hz)}} \text{ for Cu} . \]

(7.2)

(7.3)

7.2. Capacitors, inductors, and transmission lines

The capacitance between two parallel plates of area \( A \) spaced by the distance \( d \) and enclosing a medium with the dielectric constant \( \varepsilon \) is

\[ C = K \varepsilon A/d , \]  

(7.4)

where the correction factor \( K \) depends on the extent of the fringing field. If the dielectric fills the capacitor volume without extending beyond the electrodes, the correction factor \( K \approx 0.8 \) for capacitors of typical geometry.

The inductance at high frequencies of a straight wire whose length \( l \) is much greater than the wire diameter \( d \) is

\[ L \approx 2.0 \left( \frac{\ln h}{\text{cm}} \right) \cdot l \left( \ln \left( \frac{4l}{d} \right) - 1 \right) . \]  

(7.5)

For very short wires, representative of vias in a printed circuit board, the inductance is

\[ L \text{(in nH)} \approx l/d . \]  

(7.6)

A transmission line is a pair of conductors with inductance \( L \) and capacitance \( C \). The characteristic impedance is \( Z = \sqrt{L/C} \) and the phase velocity \( v_p = 1/\sqrt{LC} = 1/\sqrt{\mu_0 \varepsilon_0} \) decreases with the inverse square root of the dielectric constant of the medium. Typical coaxial and ribbon cables have a propagation delay of about 5 ns/cm.

The impedance of a coaxial cable with outer diameter \( D \) and inner diameter \( d \) is

\[ Z = 60 \, \Omega \cdot \frac{1}{\sqrt{\varepsilon_f}} \ln \frac{D}{d} . \]  

(7.7)

where the relative dielectric constant \( \varepsilon_f = \varepsilon/\varepsilon_0 \). A pair of parallel wires of diameter \( d \) and spacing \( a > 2.5 \, d \) has the impedance

\[ Z = 120 \, \Omega \cdot \frac{1}{\sqrt{\varepsilon_f}} \ln \frac{2a}{d} . \]  

(7.8)

This yields the impedance of a wire at a spacing \( h \) above a ground plane,

\[ Z = 60 \, \Omega \cdot \frac{1}{\sqrt{\varepsilon_f}} \ln \frac{4h}{d} . \]  

(7.9)

A common configuration utilizes a thin rectangular conductor above a ground plane with an intermediate dielectric (microstrip). Detailed calculations for this and other transmission line configurations are given by Gunston.*


7.3. Synchrotron radiation (CGS units)

For a particle of charge \( e \), velocity \( v = \beta c \), and energy \( E = \gamma mc^2 \), traveling in a circular orbit of radius \( R \), the classical energy loss per revolution \( \delta E \) is

\[ \delta E = \frac{4\pi e^2}{3} \frac{\gamma^3}{R} . \]  

(7.10)

For high-energy electrons or positrons \((\beta \approx 1)\), this becomes

\[ \delta E \approx 0.0885 \, \varepsilon (\text{in GeV}) \frac{E (\text{in MeV})}{R (\text{in m})} . \]  

(7.11)

For \( \gamma \gg 1 \), the energy radiated per revolution into the photon energy interval \( d(\hbar \omega) \) is

\[ dI = \frac{8\pi e^2}{9} \gamma \frac{\omega}{\omega_\gamma} \, d(\hbar \omega) , \]  

(7.12)

where \( a = e^2/\hbar c \) is the fine-structure constant and

\[ \omega_\gamma = \frac{3\gamma^3 c}{2R} , \]

is the critical frequency. The normalized function \( F(y) \) is

\[ F(y) \approx \left( \frac{9}{8\pi} \right)^{3/2} \int_y^\infty K_{5/3} (x) \, dx , \]  

(7.14)

where \( K_{5/3} (x) \) is a modified Bessel function of the third kind. For electrons or positrons, \( \hbar \omega_\gamma \approx 2.22 \, [E (\text{in GeV})]^3 / R (\text{in m}) \) .

Fig. 7.1 shows \( F(y) \) over the important range of \( y \).

---

![Figure 7.1: The normalized synchrotron radiation spectrum \( F(y) \).](image)

For \( \gamma \gg 1 \) and \( \omega \ll \omega_\gamma \),

\[ \frac{dI}{d(\hbar \omega)} \approx 3.3a (\omega R/c)^{1/3} , \]  

(7.16)

whereas for \( \gamma \gg 1 \) and \( \omega \gg 3\omega_\gamma \),

\[ \frac{dI}{d(\hbar \omega)} \approx \sqrt{\frac{3\pi}{2}} \alpha \gamma \left( \frac{\omega}{\omega_\gamma} \right)^{1/2} e^{-\omega/\omega_\gamma} \left[ 1 + \frac{55}{72} \frac{\omega_\gamma}{\omega} + \ldots \right] . \]  

(7.17)

The radiation is confined to angles \( \lesssim 1/\gamma \) relative to the instantaneous direction of motion. For \( \gamma \gg 1 \), where Eq. (7.12) applies, the mean number of photons emitted per revolution is

\[ N_\gamma = \frac{5\pi}{3} \alpha \gamma , \]  

(7.18)

and the mean energy per photon is

\[ \langle \hbar \omega \rangle = \frac{8}{15\sqrt{\gamma}} \hbar \omega_\gamma . \]  

(7.19)

When \( \langle \hbar \omega \rangle \gtrsim O(E) \), quantum corrections are important.

See J.D. Jackson, *Classical Electrodynamics*, 3rd edition (John Wiley & Sons, New York, 1998) for more formulas and details. (Note that earlier editions had \( \omega_\gamma \) twice as large as Eq. (7.13).