$V_{ud}$, $V_{us}$, THE CABIBBO ANGLE, 
AND CKM UNITARITY

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The Cabibbo-Kobayashi-Maskawa (CKM) \cite{1,2} three-generation quark mixing matrix written in terms of the Wolfenstein parameters ($\lambda, A, \rho, \eta$) \cite{3} nicely illustrates the orthonormality constraint of unitarity and central role played by $\lambda$.

$$V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \lambda^2/2 & \lambda A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \lambda^2/2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + O(\lambda^4). \quad (1)$$

That cornerstone is a carryover from the two-generation Cabibbo angle, $\lambda = \sin(\theta_{\text{Cabibbo}}) = V_{us}$. Its value is a critical ingredient in determinations of the other parameters and in tests of CKM unitarity.

Unfortunately, the precise value of $\lambda$ has been somewhat controversial in the past, with kaon decays suggesting \cite{4} $\lambda \simeq 0.220$, while hyperon decays \cite{5} and indirect determinations via nuclear $\beta$-decays imply a somewhat larger $\lambda \simeq 0.225 - 0.230$. That discrepancy is often discussed in terms of a deviation from the unitarity requirement

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1. \quad (2)$$

For many years, using a value of $V_{us}$ derived from $K \to \pi e\nu$ ($K_{e3}$) decays, that sum was consistently 2–2.5 sigma below unity, a potential signal \cite{6} for new physics effects. Below, we discuss the current status of $V_{ud}$, $V_{us}$, and their associated unitarity test in Eq. (2). (Since $|V_{ub}|^2 \approx 1 \times 10^{-5}$ is negligibly small, it is ignored in this discussion.)

$V_{ud}$

The value of $V_{ud}$ has been obtained from superallowed nuclear, neutron, and pion decays. Currently, the most precise
determination of $V_{ud}$ comes from superallowed nuclear beta-decays [6] ($0^+ \rightarrow 0^+$ transitions). Measuring their half-lives, $t$, and $Q$ values which give the decay rate factor, $f$, leads to a precise determination of $V_{ud}$ via the master formula [7–9]

$$|V_{ud}|^2 = \frac{2984.48(5) \text{ sec}}{ft(1 + RC)}$$

(3)

where RC denotes the entire effect of electroweak radiative corrections, nuclear structure, and isospin violating nuclear effects. RC is nucleus-dependent, ranging from about $+3.0\%$ to $+3.6\%$ for the nine best measured superallowed decays. In Table 1, we give updated [10] $ft$ values along with their implied $V_{ud}$ for the nine best measured superallowed decays [6, 10]. They collectively give a weighted average (with errors combined in quadrature) of

$$V_{ud} = 0.97418(27) \text{ (superallowed)} ,$$

(4)

which, assuming unitarity, corresponds to $\lambda = 0.226(1)$. We note that the new average value of $V_{ud}$ is shifted upward compared to our 2005 value of 0.97377(27) primarily because of a recent reevaluation of the isospin breaking Coulomb corrections by Towner and Hardy [10].

Combined measurements of the neutron lifetime, $\tau_n$, and the ratio of axial-vector/vector couplings, $g_A \equiv G_A/G_V$, via neutron decay asymmetries can also be used to determine $V_{ud}$:

$$|V_{ud}|^2 = \frac{4908.7(1.9) \text{ sec}}{\tau_n(1 + 3g_A^2)},$$

(5)

where the error stems from uncertainties in the electroweak radiative corrections [8] due to hadronic loop effects. Those effects have been recently updated and their error was reduced by about a factor of 2 [9], leading to a $\pm 0.0002$ theoretical uncertainty in $V_{ud}$ (common to all $V_{ud}$ extractions). Using the world averages from this Review

$$\tau_n^{\text{ave}} = 885.7(8) \text{ sec}$$

$$g_A^{\text{ave}} = 1.2695(29)$$

(6)
Table 1: Values of $V_{ud}$ implied by various precisely measured superallowed nuclear beta decays. The $ft$ values and Coulomb isospin breaking corrections are taken from Towner and Hardy [10]. Uncertainties in $V_{ud}$ correspond to 1) nuclear structure and $Z^2\alpha^3$ uncertainties [6, 11] added in quadrature with the $ft$ error; 2) a common error assigned to nuclear Coulomb distortion effects [11]; and 3) a common uncertainty in the radiative corrections from quantum loop effects [9]. Only the first error is used to obtain the weighted average.

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>$ft$ (sec)</th>
<th>$V_{ud}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{10}C$</td>
<td>3039.5(47)</td>
<td>0.97370(80)(14)(19)</td>
</tr>
<tr>
<td>$^{14}O$</td>
<td>3042.5(27)</td>
<td>0.97411(51)(14)(19)</td>
</tr>
<tr>
<td>$^{26}Al$</td>
<td>3037.0(11)</td>
<td>0.97400(24)(14)(19)</td>
</tr>
<tr>
<td>$^{34}Cl$</td>
<td>3050.0(11)</td>
<td>0.97417(34)(14)(19)</td>
</tr>
<tr>
<td>$^{38}K$</td>
<td>3051.1(10)</td>
<td>0.97413(39)(14)(19)</td>
</tr>
<tr>
<td>$^{42}Sc$</td>
<td>3046.4(14)</td>
<td>0.97423(44)(14)(19)</td>
</tr>
<tr>
<td>$^{46}V$</td>
<td>3049.6(16)</td>
<td>0.97386(49)(14)(19)</td>
</tr>
<tr>
<td>$^{50}Mn$</td>
<td>3044.4(12)</td>
<td>0.97487(45)(14)(19)</td>
</tr>
<tr>
<td>$^{54}Co$</td>
<td>3047.6(15)</td>
<td>0.97490(54)(14)(19)</td>
</tr>
</tbody>
</table>

Weighted Ave. 0.97418(13)(14)(19)

leads to

$$V_{ud} = 0.9746(4)\tau_n(18)g_A(2)_{RC}$$ (7)

with the error dominated by $g_A$ uncertainties (which have been expanded due to experimental inconsistencies). We note that a recent precise measurement [12] of $\tau_n = 878.5(7)(3)$ sec is also inconsistent with the world average from this Review and would lead to a considerably larger $V_{ud} = 0.9786(4)(18)(2)$. Future neutron studies are expected to resolve these inconsistencies and significantly reduce the uncertainties in $g_A$ and $\tau_n$, potentially making them the best way to determine $V_{ud}$.

The recently completed PIBETA experiment at PSI measured the very small ($O(10^{-8})$) branching ratio for $\pi^+ \rightarrow \pi^0 e^+\nu_e$ with about ±1/2% precision. Their result gives [13]
\[ V_{ud} = 0.9749(26) \left( \frac{BR(\pi^+ \rightarrow e^+\bar{\nu}_e(\gamma))}{1.2352 \times 10^{-4}} \right)^{1/2} \]  

which is normalized using the very precisely determined theoretical prediction for \( BR(\pi^+ \rightarrow e^+\bar{\nu}_e(\gamma)) = 1.2352(5) \times 10^{-4} [7], \) rather than the experimental branching ratio from this Review of \( 1.230(4) \times 10^{-4} \) which would lower the value to \( V_{ud} = 0.9728(30). \) Theoretical uncertainties in that determination are very small; however, much higher statistics would be required to make this approach competitive with others.

**\( V_{us} \)**

\(|V_{us}|\) may be determined from kaon decays, hyperon decays, and tau decays. Previous determinations have most often used \( K\ell 3 \) decays:

\[ \Gamma_{K\ell 3} = \frac{G_F^2 M_K^5}{192\pi^3} S_{EW}(1 + \delta_K + \delta_{SU2})C^2 |V_{us}|^2 f_+(0) I_{K}. \]  

Here, \( \ell \) refers to either \( e \) or \( \mu, \) \( G_F \) is the Fermi constant, \( M_K \) is the kaon mass, \( S_{EW} \) is the short-distance radiative correction, \( \delta_K \) is the mode-dependent long-distance radiative correction, \( f_+(0) \) is the calculated form factor at zero momentum transfer for the \( \ell\nu \) system, and \( I_{K}^{\ell} \) is the phase-space integral, which depends on measured semileptonic form factors. For charged kaon decays, \( \delta_{SU2} \) is the deviation from one of the ratio of \( f_+(0) \) for the charged to neutral kaon decay; it is zero for the neutral kaon. \( C^2 \) is 1 (1/2) for neutral (charged) kaon decays. Most determinations of \( |V_{us}| \) have been based only on \( K \rightarrow \pi\ell\nu \) decays; \( K \rightarrow \pi\mu\nu \) decays have not been used because of large uncertainties in \( I_{K}^{\mu} \). The experimental measurements are the semileptonic decay widths (based on the semileptonic branching fractions and lifetime) and form factors (allowing calculation of the phase space integrals). Theory is needed for \( S_{EW}, \delta_K, \delta_{SU2}, \) and \( f_+(0). \)

Many new measurements during the last few years have resulted in a significant shift in \( V_{us}. \) Most importantly, recent measurements of the \( K \rightarrow \pi\ell\nu \) branching fractions are significantly different than earlier PDG averages, probably as
a result of inadequate treatment of radiation in older experiments. This effect was first observed by BNL E865 [14] in the charged kaon system and then by KTeV [15,16] in the neutral kaon system; subsequent measurements were made by KLOE [17–20], NA48 [21–23], and ISTRA+ [24]. Current averages (e.g., by the PDG [25] or Flavianet [26]) of the semileptonic branching fractions are based only on recent, high-statistics experiments where the treatment of radiation is clear. In addition to measurements of branching fractions, new measurements of lifetimes [27] and form factors [28–32], have resulted in improved precision for all of the experimental inputs to \( V_{us} \). Precise measurements of form factors for \( K_{\mu 3} \) decay now make it possible to use both semileptonic decay modes to extract \( V_{us} \).

Following the analysis of the Flavianet group [26], one finds the values of \( |V_{us}|f_+(0) \) in Table 2. The average of these measurements gives

\[
f_+(0)|V_{us}| = 0.21668(45).
\]  

(10)

Figure 1 shows a comparison of these results with the PDG evaluation from 2002 [33], as well as \( f_+(0)(1-|V_{ud}|^2-|V_{ub}|^2)^{1/2} \), the expectation for \( f_+(0)|V_{us}| \) assuming unitarity, based on \( |V_{ud}| = 0.9742\pm 0.0003, |V_{ub}| = (3.6\pm 0.7) \times 10^{-3} \), and the widely used Leutwyler-Roos calculation of \( f_+(0) = 0.961 \pm 0.008 \) [34].

Using the result in Eq. (10) with the Leutwyler-Roos calculation of \( f_+(0) \) gives

\[
|V_{us}| = \lambda = 0.2255 \pm 0.0019.
\]  

(11)

Similar results for \( f_+(0) \) were recently obtained from lattice gauge theory calculations [35,36]. For example, and recent 2+1 fermion dynamical wall calculation [36] gave \( f_+(0) = 0.9609(51) \). Other calculations of \( f_+(0) \) result in \( |V_{us}| \) values that differ by as much as 2% from the result in Eq. (11). For example, a recent chiral perturbation theory calculation [37, 38] gives \( f_+(0) = 0.974 \pm 0.012 \), which implies a lower value of \( |V_{us}| = 0.2225 \pm 0.0028 \) [39].
Table 2: \(|V_{us}|f_+(0)\) from \(K_{\ell 3}\).

| Decay Mode | \(|V_{us}|f_+(0)\) |
|------------|-------------------|
| \(K^{\pm}\ell3\) | 0.21746 ± 0.00085 |
| \(K^{\pm}\mu 3\) | 0.21810 ± 0.00114 |
| \(K_L\ell3\) | 0.21638 ± 0.00055 |
| \(K_L\mu 3\) | 0.21678 ± 0.00067 |
| \(K_S\ell3\) | 0.21554 ± 0.00142 |
| **Average** | 0.21668 ± 0.00045 |

Figure 1: Comparison of determinations of \(|V_{us}|f_+(0)\) from this review (labeled 2005), from the PDG 2002, and with the prediction from unitarity using \(|V_{ud}|\) and the Leutwyler-Roos calculation of \(f_+(0)\) [34]. For \(f_+(0)(1-|V_{ud}|^2-|V_{ub}|^2)^{1/2}\), the inner error bars are from the quoted uncertainty in \(f_+(0)\); the total uncertainties include the \(|V_{ud}|\) and \(|V_{ub}|\) errors. See full-color version on color pages at end of book.

A value of \(V_{us}\) can also be obtained from a comparison of the radiative inclusive decay rates for \(K \rightarrow \ell \nu \gamma\) and \(\pi \rightarrow \mu \nu \gamma\) combined with a lattice gauge theory calculation of \(f_K/f_\pi\) via [40]
\[ \frac{|V_{us}| f_K}{|V_{ud}| f_\pi} = 0.2387(4) \left[ \frac{\Gamma(K \to \mu \nu(\gamma))}{\Gamma(\pi \to \mu \nu(\gamma))} \right]^{\frac{1}{2}} \]  
(12)

with the small error coming from electroweak radiative corrections. Employing

\[ \frac{\Gamma(K \to \mu \nu(\gamma))}{\Gamma(\pi \to \mu \nu(\gamma))} = 1.3337(46), \]  
(13)

which averages in the KLOE result \[41\], \[B(K \to \mu \nu(\gamma)) = 63.66(9)(15)\%\] and \[42, 43\]

\[ f_K/f_\pi = 1.208(2)(+7/−14) \]  
(14)

along with the value of \(V_{ud}\) in Eq. (4) leads to

\[ |V_{us}| = 0.2223(5)(1.208 f_\pi/f_K). \]  
(15)

It should be mentioned that hyperon decay fits suggest \[5\]

\[ |V_{us}| = 0.2250(27) \]  
(16)

modulo SU(3) breaking effects that could shift that value up or down. We note that a recent representative effort \[44\] that incorporates SU(3) breaking found \(V_{us} = 0.226(5)\). Similarly, strangeness changing tau decays give \[45\]

\[ |V_{us}| = 0.2208(34) \]  
(17)

where the central value depends on the strange quark mass.

Employing the value of \(V_{ud}\) in Eq. (4) and \(V_{us}\) in Eq. (11)
leads to the unitarity consistency check

\[ |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9999(5)(9), \]  
(18)

where the first error is the uncertainty from \(|V_{ud}|^2\) and the second error is the uncertainty from \(|V_{us}|^2\). The result is in good agreement with unitarity. Averaging the direct determination of \(\lambda (V_{us})\) with the determination derived from unitarity and \(V_{ud}\) gives \(\lambda = 0.226(1)\). Although unitarity now seems well established, issues regarding the Q values in superallowed nuclear \(\beta\)-decays, \(\tau_n, g_A, f_+(0)\) and \(f_K/f_\pi\) must still be resolved before a definitive confirmation is possible.
CKM Unitarity Constraints

The current good experimental agreement with unitarity, $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9999(10)$ provides strong confirmation of Standard Model radiative corrections (which range between 3-4% depending on the nucleus used) at better than the 30 sigma level [46]. In addition, it implies constraints on “New Physics” effects at both the tree and quantum loop levels. Those effects could be in the form of contributions to nuclear beta decays, $K_{e3}$ decays and/or muon decays, with the last of these providing normalization via the muon lifetime [47], which is used to obtain the Fermi constant, $G_\mu = 1.166371(6) \times 10^{-5}$GeV$^{-2}$.

We illustrate the implications of CKM unitarity for: 1) exotic muon decays [48]( beyond ordinary muon decay $\mu^+ \rightarrow e^+\nu_e\bar{\nu}_\mu$); and 2) new heavy quark mixing $V_{ud}$ [49]. Other examples in the literature [50,51] include $Z_\chi$ boson quantum loop effects, supersymmetry, leptoquarks, compositeness etc.

Exotic Muon Decays

If additional lepton flavor violating decays such as $\mu^+ \rightarrow e^+\bar{\nu}_e\nu_\mu$ (wrong neutrinos) occur, they would cause confusion in searches for neutrino oscillations at, for example, muon storage rings/neutrino factories or other neutrino sources from muon decays. Calling the rate for all such decays $\Gamma$(exotic $\mu$ decays), they should be subtracted before the extraction of $G_\mu$ and normalization of the CKM matrix. Since that is not done and unitarity works, one has (at one-sided 95% CL)

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 - BR(\text{exotic } \mu \text{ decays}) \geq 0.9982$$  \hspace{1cm} (19)

or

$$BR(\text{exotic } \mu \text{ decays}) < 0.0018.$$  \hspace{1cm} (20)

That bound is a factor of 6–7 better than the direct experimental bound on $\mu^+ \rightarrow e^+\bar{\nu}_e\nu_\mu$.

New Heavy Quark Mixing
Heavy $D$ quarks naturally occur in fourth quark generation models and some heavy quark “new physics” scenarios such as $E_6$ grand unification. Their mixing with ordinary quarks gives rise to $V_{ud}$ which is constrained by unitarity (one sided 95% CL)

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 - |V_{uD}|^2 > 0.9982$$
$$|V_{uD}| < 0.04 .$$ (21)

A similar constraint applies to heavy neutrino mixing and the couplings $V_{\mu N}$ and $V_{eN}$.

References

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