## 7. ELECTROMAGNETIC RELATIONS

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Quantity	Gaussian CGS	SI
Conversion factors:		
Charge:	$2.99792458 \times 10^9 \text{ esu}$	= 1  C = 1  A s
Potential:	(1/299.792458) statvolt $(ergs/esu)$	$= 1 \text{ V} = 1 \text{ J C}^{-1}$
Magnetic field:	$10^4 \text{ gauss} = 10^4 \text{ dyne/esu}$	$= 1 \text{ T} = 1 \text{ N A}^{-1} \text{m}^{-1}$
	$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)$	$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$
	$\nabla \cdot \mathbf{D} = 4\pi \rho$	$\nabla \cdot \mathbf{D} = \rho$
	$\nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} = \frac{4\pi}{c} \mathbf{J}$	$\mathbf{\nabla}  imes \mathbf{H} - rac{\partial \mathbf{D}}{\partial t} = \mathbf{J}$
	$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$
	$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0$	$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$
Constitutive relations:	$\mathbf{D} = \mathbf{E} + 4\pi \mathbf{P},  \mathbf{H} = \mathbf{B} - 4\pi \mathbf{M}$	$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P},  \mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$
Linear media:	$\mathbf{D} = \epsilon \mathbf{E},  \mathbf{H} = \mathbf{B}/\mu$	$\mathbf{D} = \epsilon \mathbf{E},  \mathbf{H} = \mathbf{B}/\mu$
	1	$\epsilon_0 = 8.854 \ 187 \dots \times 10^{-12} \ \mathrm{F m}^{-1}$
	1	$\mu_0 = 4\pi \times 10^{-7} \text{ N A}^{-2}$
	$\mathbf{E} = -\nabla V - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$	$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$
	$\mathbf{B} = \mathbf{\nabla} \times \mathbf{A}$	$\mathbf{B} = \mathbf{\nabla} \times \mathbf{A}$
	$V = \sum_{\text{charges}} \frac{q_i}{r_i} = \int \frac{\rho(\mathbf{r}')}{ \mathbf{r} - \mathbf{r}' } d^3x'$	$V = \frac{1}{4\pi\epsilon_0} \sum_{\text{charges}} \frac{q_i}{r_i} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{ \mathbf{r} - \mathbf{r}' } d^3 x'$
	$\mathbf{A} = \frac{1}{c} \oint \frac{I  d\ell}{ \mathbf{r} - \mathbf{r}' } = \frac{1}{c} \int \frac{\mathbf{J}(\mathbf{r}')}{ \mathbf{r} - \mathbf{r}' }  d^3 x'$	$\mathbf{A} = \frac{\mu_0}{4\pi} \oint \frac{I  \mathbf{d}\boldsymbol{\ell}}{ \mathbf{r} - \mathbf{r}' } = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{ \mathbf{r} - \mathbf{r}' }  d^3 x'$
	$\mathbf{E}_{\parallel}' = \mathbf{E}_{\parallel}$	$\mathbf{E}_{\parallel}' = \mathbf{E}_{\parallel}$
	$\mathbf{E}'_{\perp} = \gamma (\mathbf{E}_{\perp} + \frac{1}{c} \mathbf{v} \times \mathbf{B})$	$\mathbf{E}'_{\perp} = \gamma (\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B})$
	$\mathbf{B}_{\parallel}'=\mathbf{B}_{\parallel}$	$\mathbf{B}_{\parallel}'=\mathbf{B}_{\parallel}$
	$\mathbf{B}_{\perp}' = \gamma (\mathbf{B}_{\perp} - \frac{1}{c} \mathbf{v} \times \mathbf{E})$	$\mathbf{B}'_{\perp} = \gamma (\mathbf{B}_{\perp} - \frac{1}{c^2} \mathbf{v} \times \mathbf{E})$
$\frac{1}{4\pi\epsilon_0} = c^2 \times 10^{-7} \text{ N A}^{-2} = 8.98755 \times 10^9 \text{ m F}^{-1} ; \frac{\mu_0}{4\pi} = 10^{-7} \text{ N A}^{-2} ; c = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 2.99792458 \times 10^8 \text{ m s}^{-1}$		

## 7.1. Impedances (SI units)

 $\rho = \text{resistivity at room temperature in } 10^{-8} \,\Omega \text{ m}$ :

For alternating currents, instantaneous current I, voltage V, angular frequency  $\omega$ :

$$V = V_0 e^{j\omega t} = ZI . (7.1)$$

Impedance of self-inductance L:  $Z = j\omega L$ .

Impedance of capacitance  $C\colon\thinspace Z=1/j\omega C$  .

Impedance of free space:  $Z = \sqrt{\mu_0/\epsilon_0} = 376.7 \ \Omega$ .

High-frequency surface impedance of a good conductor:

$$Z = \frac{(1+j) \rho}{\delta} , \quad \text{where } \delta = \text{skin depth} ;$$
 (7.2)

$$\delta = \sqrt{\frac{\rho}{\pi \nu \mu}} \approx \frac{6.6 \text{ cm}}{\sqrt{\nu \text{ (Hz)}}} \text{ for Cu}.$$
 (7.3)

## 7.2. Capacitors, inductors, and transmission Lines

The capacitance between two parallel plates of area A spaced by the distance d and enclosing a medium with the dielectric constant  $\varepsilon$  is

$$C = K\varepsilon A/d, \tag{7.4}$$

where the correction factor K depends on the extent of the fringing field. If the dielectric fills the capacitor volume without extending beyond the electrodes. the correction factor  $K\approx 0.8$  for capacitors of typical geometry.

The inductance at high frequencies of a straight wire whose length  $\ell$  is much greater than the wire diameter d is

$$L \approx 2.0 \left[ \frac{\mathrm{nH}}{\mathrm{cm}} \right] \cdot \ell \left( \ln \left( \frac{4\ell}{d} \right) - 1 \right)$$
 (7.5)

For very short wires, representative of vias in a printed circuit board, the inductance is

$$L(\text{in nH}) \approx \ell/d$$
. (7.6)

A transmission line is a pair of conductors with inductance L and capacitance C. The characteristic impedance  $Z=\sqrt{L/C}$  and the phase velocity  $v_p=1/\sqrt{LC}=1/\sqrt{\mu\varepsilon}$ , which decreases with the inverse square root of the dielectric constant of the medium. Typical coaxial and ribbon cables have a propagation delay of about  $5\,\mathrm{ns/cm}$ . The impedance of a coaxial cable with outer diameter D and inner diameter d is

$$Z = 60 \Omega \cdot \frac{1}{\sqrt{\varepsilon_r}} \ln \frac{D}{d}, \qquad (7.7)$$

where the relative dielectric constant  $\varepsilon_r = \varepsilon/\varepsilon_0$ . A pair of parallel wires of diameter d and spacing  $a>2.5\,d$  has the impedance

$$Z = 120 \,\Omega \cdot \frac{1}{\sqrt{\varepsilon_r}} \ln \frac{2a}{d} \,. \tag{7.8}$$

This yields the impedance of a wire at a spacing h above a ground plane.

$$Z = 60 \Omega \cdot \frac{1}{\sqrt{\varepsilon_r}} \ln \frac{4h}{d}. \tag{7.9}$$

A common configuration utilizes a thin rectangular conductor above a ground plane with an intermediate dielectric (microstrip). Detailed calculations for this and other transmission line configurations are given by Gunston.\*

## 7.3. Synchrotron radiation (CGS units)

For a particle of charge e, velocity  $v = \beta c$ , and energy  $E = \gamma mc^2$ , traveling in a circular orbit of radius R, the classical energy loss per revolution  $\delta E$  is

$$\delta E = \frac{4\pi}{3} \, \frac{e^2}{R} \, \beta^3 \, \gamma^4 \, . \tag{7.10}$$

For high-energy electrons or positrons ( $\beta \approx 1$ ), this becomes

$$\delta E \text{ (in MeV)} \approx 0.0885 \ [E(\text{in GeV})]^4 / R(\text{in m}) \ .$$
 (7.11)

For  $\gamma\gg 1$ , the energy radiated per revolution into the photon energy interval  $d(\hbar\omega)$  is

$$dI = \frac{8\pi}{9} \alpha \gamma F(\omega/\omega_c) d(\hbar\omega) , \qquad (7.12)$$

where  $\alpha = e^2/\hbar c$  is the fine-structure constant and

$$\omega_c = \frac{3\gamma^3 c}{2R} \tag{7.13}$$

is the critical frequency. The normalized function F(y) is

$$F(y) = \frac{9}{8\pi} \sqrt{3} y \int_{y}^{\infty} K_{5/3}(x) dx$$
, (7.14)

where  $K_{5/3}\left(x\right)$  is a modified Bessel function of the third kind. For electrons or positrons,

$$\hbar\omega_c \,(\text{in keV}) \approx 2.22 \,\left[E(\text{in GeV})\right]^3 / R(\text{in m}) \,.$$
 (7.15)

Fig. 7.1 shows F(y) over the important range of y.

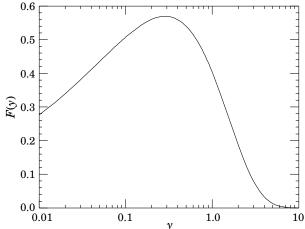


Figure 7.1: The normalized synchrotron radiation spectrum F(y).

For  $\gamma \gg 1$  and  $\omega \ll \omega_c$ ,

$$\frac{dI}{d(\hbar\omega)} \approx 3.3\alpha \left(\omega R/c\right)^{1/3} ,$$
 (7.16)

whereas for

$$\gamma \gg 1$$
 and  $\omega \gtrsim 3\omega_c$ 

$$\frac{dI}{d(\hbar\omega)} \approx \sqrt{\frac{3\pi}{2}} \, \alpha \, \gamma \left(\frac{\omega}{\omega_c}\right)^{1/2} e^{-\omega/\omega_c} \left[1 + \frac{55}{72} \frac{\omega_c}{\omega} + \ldots\right] \quad . \tag{7.17}$$

The radiation is confined to angles  $\lesssim 1/\gamma$  relative to the instantaneous direction of motion. For  $\gamma \gg 1$ , where Eq. (7.12) applies, the mean number of photons emitted per revolution is

$$N_{\gamma} = \frac{5\pi}{\sqrt{3}} \alpha \gamma , \qquad (7.18)$$

and the mean energy per photon is

$$\langle \hbar \omega \rangle = \frac{8}{15\sqrt{3}} \hbar \omega_c \ . \tag{7.19}$$

When  $\langle \hbar \omega \rangle \gtrsim \mathcal{O}(E)$ , quantum corrections are important.

See J.D. Jackson, Classical Electrodynamics,  $3^{\rm rd}$  edition (John Wiley & Sons, New York, 1998) for more formulae and details. (Note that earlier editions had  $\omega_c$  twice as large as Eq. (7.13).

<sup>\*</sup> M.A.R. Gunston. Microwave Transmission Line Data, Noble Publishing Corp., Atlanta (1997) ISBN 1-884932-57-6, TK6565.T73G85.