46. KINEMATICS

Revised January 2000 by J.D. Jackson (LBNL) and June 2008 and January 2016 by D.R. Tovey (Sheffield).

Throughout this section units are used in which $\hbar = c = 1$. The following conversions are useful: $\hbar c = 197.3$ MeV fm, $(\hbar c)^2 = 0.3894$ $(\mathrm{GeV})^2$ mb.

46.1. Lorentz transformations

The energy E and 3-momentum $\boldsymbol{p}$ of a particle of mass m form a 4-vector $p = (E, \boldsymbol{p})$ whose square $p^2 \equiv E^2 - |\boldsymbol{p}|^2 = m^2$. The velocity of the particle is $\boldsymbol{\beta} = \boldsymbol{p}/E$. The energy and momentum $(E^*, \boldsymbol{p}^*)$ viewed from a frame moving with velocity $\boldsymbol{\beta}_f$ are given by

$$\begin{pmatrix} E^* \\ p_\parallel^* \end{pmatrix} = \begin{pmatrix} \gamma_f & -\gamma_f\beta_f \\ -\gamma_f\beta_f & \gamma_f \end{pmatrix} \begin{pmatrix} E \\ p_\parallel \end{pmatrix} , \qquad p_T^* = p_T , \tag{46.1}$$

where $\gamma_f = (1 - \beta_f^2)^{-1/2}$ and p_T $(p_\parallel)$ are the components of $\boldsymbol{p}$ perpendicular (parallel) to $\boldsymbol{\beta}_f$. Other 4-vectors, such as the space-time coordinates of events, of course transform in the same way. The scalar product of two 4-momenta $p_1 \cdot p_2 = E_1 E_2 - \boldsymbol{p}_1 \cdot \boldsymbol{p}_2$ is invariant (frame independent).

46.2. Center-of-mass energy and momentum

In the collision of two particles of masses m_1 and m_2 the total center-of-mass energy can be expressed in the Lorentz-invariant form

$$\begin{aligned} E_{\mathrm{cm}} &= \Big[(E_1 + E_2)^2 - (\boldsymbol{p}_1 + \boldsymbol{p}_2)^2 \Big]^{1/2} , \\ &= \Big[m_1^2 + m_2^2 + 2E_1 E_2 (1 - \beta_1\beta_2 \cos\theta) \Big]^{1/2} , \end{aligned} \tag{46.2}$$

where θ is the angle between the particles. In the frame where one particle (of mass m_2) is at rest (lab frame),

$$E_{\mathrm{cm}} = (m_1^2 + m_2^2 + 2E_{1\,\mathrm{lab}}\, m_2)^{1/2} . \tag{46.3}$$

The velocity of the center-of-mass in the lab frame is

$$\boldsymbol{\beta}_{\mathrm{cm}} = \boldsymbol{p}_{\mathrm{lab}}/(E_{1\,\mathrm{lab}} + m_2) , \tag{46.4}$$

where $\boldsymbol{p}_{\mathrm{lab}} \equiv \boldsymbol{p}_{1\,\mathrm{lab}}$ and

$$\gamma_{\mathrm{cm}} = (E_{1\,\mathrm{lab}} + m_2)/E_{\mathrm{cm}} . \tag{46.5}$$

The c.m. momenta of particles 1 and 2 are of magnitude

$$p_{\mathrm{cm}} = p_{\mathrm{lab}} \frac{m_2}{E_{\mathrm{cm}}} . \tag{46.6}$$

For example, if a 0.80 GeV/c kaon beam is incident on a proton target, the center of mass energy is 1.699 GeV and the center of mass momentum of either particle is 0.442 GeV/c. It is also useful to note that

$$E_{\mathrm{cm}}\, dE_{\mathrm{cm}} = m_2\, dE_{1\,\mathrm{lab}} = m_2\, \beta_{1\,\mathrm{lab}}\, dp_{\mathrm{lab}} . \tag{46.7}$$

46.3. Lorentz-invariant amplitudes

The matrix elements for a scattering or decay process are written in terms of an invariant amplitude $-i\mathscr{M}$. As an example, the S-matrix for $2 \to 2$ scattering is related to $\mathscr{M}$ by

$$\begin{aligned} \langle p_1' p_2' \, |S| \, p_1 p_2 \rangle = & \, I - i(2\pi)^4 \, \delta^4(p_1 + p_2 - p_1' - p_2') \\ & \times \frac{\mathscr{M}(p_1, \, p_2; \, p_1', \, p_2')}{(2E_1)^{1/2} \, (2E_2)^{1/2} \, (2E_1')^{1/2} \, (2E_2')^{1/2}} \; . \end{aligned} \tag{46.8}$$

The state normalization is such that

$$\langle p' | p \rangle = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{p}') \; . \tag{46.9}$$

For a $2 \to 2$ scattering process producing unstable particles $1'$ and $2'$ decaying via $1' \to 3'4'$ and $2' \to 5'6'$ the matrix element for the complete process can be written in the narrow width approximation as:

$$\mathscr{M}(12 \to 3'4'5'6') = \sum_{h_{1'}, h_{2'}} \frac{\mathscr{M}(12 \to 1'2') \mathscr{M}(1' \to 3'4') \mathscr{M}(2' \to 5'6')}{(m_{3'4'}^2 - m_{1'}^2 + i m_{1'} \Gamma_{1'})(m_{5'6'}^2 - m_{2'}^2 + i m_{2'} \Gamma_{2'})}. \tag{46.10}$$

Here, m_{ij} is the invariant mass of particles i and j, m_k and Γ_k are the mass and total width of particle k, and the sum runs over the helicities of the intermediate particles. This enables the cross section for such a process to be written as the product of the cross section for the initial $2 \to 2$ scattering process with the branching ratios (relative partial decay rates) of the subsequent decays.

46.4. Particle decays

The partial decay rate of a particle of mass M into n bodies in its rest frame is given in terms of the Lorentz-invariant matrix element $\mathscr{M}$ by

$$d\Gamma = \frac{(2\pi)^4}{2M} \, |\mathscr{M}|^2 \, d\Phi_n \, (P; \, p_1, \, \ldots, \, p_n), \tag{46.11}$$

where $d\Phi_n$ is an element of n-body phase space given by

$$d\Phi_n(P; \, p_1, \, \ldots, \, p_n) = \delta^4 \, (P - \sum_{i=1}^{n} p_i) \, \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2E_i} \; . \tag{46.12}$$

This phase space can be generated recursively, viz.

$$\begin{aligned} d\Phi_n(P; \, p_1, \, \ldots, \, p_n) = & \, d\Phi_j(q; \, p_1, \, \ldots, \, p_j) \\ & \times \; d\Phi_{n-j+1} \, (P; \, q, \, p_{j+1}, \, \ldots, \, p_n)(2\pi)^3 dq^2 \; , \end{aligned} \tag{46.13}$$

where $q^2 = (\sum_{i=1}^{j} E_i)^2 - \left| \sum_{i=1}^{j} \boldsymbol{p}_i \right|^2$. This form is particularly useful in the case where a particle decays into another particle that subsequently decays.

46.4.1. ***Survival probability*** : If a particle of mass M has mean proper lifetime τ ($= 1/\Gamma$) and has momentum $(E, \boldsymbol{p})$, then the probability that it lives for a time t_0 or greater before decaying is given by

$$P(t_0) = e^{-t_0\,\Gamma/\gamma} = e^{-M t_0\,\Gamma/E} \;, \tag{46.14}$$

and the probability that it travels a distance x_0 or greater is

$$P(x_0) = e^{-M x_0\,\Gamma/|\boldsymbol{p}|} \;. \tag{46.15}$$

46.4.2. ***Two-body decays*** :

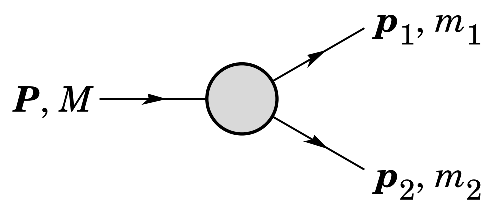

Figure 46.1: Definitions of variables for two-body decays.

In the rest frame of a particle of mass M, decaying into 2 particles labeled 1 and 2,

$$E_1 = \frac{M^2 - m_2^2 + m_1^2}{2M} \;, \tag{46.16}$$

$$|\boldsymbol{p}_1| = |\boldsymbol{p}_2|$$
$$= \frac{\left[\left(M^2 - (m_1 + m_2)^2\right)\left(M^2 - (m_1 - m_2)^2\right)\right]^{1/2}}{2M} \;, \tag{46.17}$$

and

$$d\Gamma = \frac{1}{32\pi^2}\, |\mathscr{M}|^2\, \frac{|\boldsymbol{p}_1|}{M^2}\, d\Omega \;, \tag{46.18}$$

where $d\Omega = d\phi_1 d(\cos\theta_1)$ is the solid angle of particle 1. The invariant mass M can be determined from the energies and momenta using Eq. (46.2) with $M = E_{\rm cm}$.

46.4.3. *Three-body decays* :

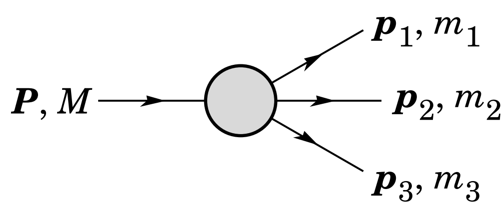

Figure 46.2: Definitions of variables for three-body decays.

Defining $p_{ij} = p_i + p_j$ and $m_{ij}^2 = p_{ij}^2$, then $m_{12}^2 + m_{23}^2 + m_{13}^2 = M^2 + m_1^2 + m_2^2 + m_3^2$ and $m_{12}^2 = (P - p_3)^2 = M^2 + m_3^2 - 2ME_3$, where E_3 is the energy of particle 3 in the rest frame of M. In that frame, the momenta of the three decay particles lie in a plane. The relative orientation of these three momenta is fixed if their energies are known. The momenta can therefore be specified in space by giving three Euler angles (α, β, γ) that specify the orientation of the final system relative to the initial particle [1]. Then

$$d\Gamma = \frac{1}{(2\pi)^5} \frac{1}{16M} \, |\mathscr{M}|^2 \, dE_1 \, dE_3 \, d\alpha \, d(\cos\beta) \, d\gamma \ . \tag{46.19}$$

Alternatively

$$d\Gamma = \frac{1}{(2\pi)^5} \frac{1}{16M^2} \, |\mathscr{M}|^2 \, |\boldsymbol{p}_1^*| \, |\boldsymbol{p}_3| \, dm_{12} \, d\Omega_1^* \, d\Omega_3 \ , \tag{46.20}$$

where $(|\boldsymbol{p}_1^*|, \ \Omega_1^*)$ is the momentum of particle 1 in the rest frame of 1 and 2, and Ω_3 is the angle of particle 3 in the rest frame of the decaying particle. $|\boldsymbol{p}_1^*|$ and $|\boldsymbol{p}_3|$ are given by

$$|\boldsymbol{p}_1^*| = \frac{\left[\left(m_{12}^2 - (m_1 + m_2)^2\right)\left(m_{12}^2 - (m_1 - m_2)^2\right)\right]^{1/2}}{2m_{12}} \ , \tag{46.21a}$$

and

$$|\boldsymbol{p}_3| = \frac{\left[\left(M^2 - (m_{12} + m_3)^2\right)\left(M^2 - (m_{12} - m_3)^2\right)\right]^{1/2}}{2M} \ . \tag{46.21b}$$

[Compare with Eq. (46.17).]

If the decaying particle is a scalar or we average over its spin states, then integration over the angles in Eq. (46.19) gives

$$\begin{aligned} d\Gamma &= \frac{1}{(2\pi)^3} \frac{1}{8M} \, \overline{|\mathscr{M}|^2} \, dE_1 \, dE_3 \\ &= \frac{1}{(2\pi)^3} \frac{1}{32M^3} \, \overline{|\mathscr{M}|^2} \, dm_{12}^2 \, dm_{23}^2 \ . \end{aligned} \tag{46.22}$$

This is the standard form for the Dalitz plot.

46.4.3.1. *Dalitz plot:* For a given value of m_{12}^2, the range of m_{23}^2 is determined by its values when $\boldsymbol{p}_2$ is parallel or antiparallel to $\boldsymbol{p}_3$:

$$(m_{23}^2)_{\rm max} = (E_2^* + E_3^*)^2 - \left(\sqrt{E_2^{*2} - m_2^2} - \sqrt{E_3^{*2} - m_3^2}\right)^2 , \quad (46.23a)$$

$$(m_{23}^2)_{\rm min} = (E_2^* + E_3^*)^2 - \left(\sqrt{E_2^{*2} - m_2^2} + \sqrt{E_3^{*2} - m_3^2}\right)^2 . \quad (46.23b)$$

Here $E_2^* = (m_{12}^2 - m_1^2 + m_2^2)/2m_{12}$ and $E_3^* = (M^2 - m_{12}^2 - m_3^2)/2m_{12}$ are the energies of particles 2 and 3 in the m_{12} rest frame. The scatter plot in m_{12}^2 and m_{23}^2 is called a Dalitz plot. If $\overline{|\mathscr{M}|^2}$ is constant, the allowed region of the plot will be uniformly populated with events [see Eq. (46.22)]. A nonuniformity in the plot gives immediate information on $|\mathscr{M}|^2$. For example, in the case of $D \to K\pi\pi$, bands appear when $m_{(K\pi)} = m_{K^*(892)}$, reflecting the appearance of the decay chain $D \to K^*(892)\pi \to K\pi\pi$.

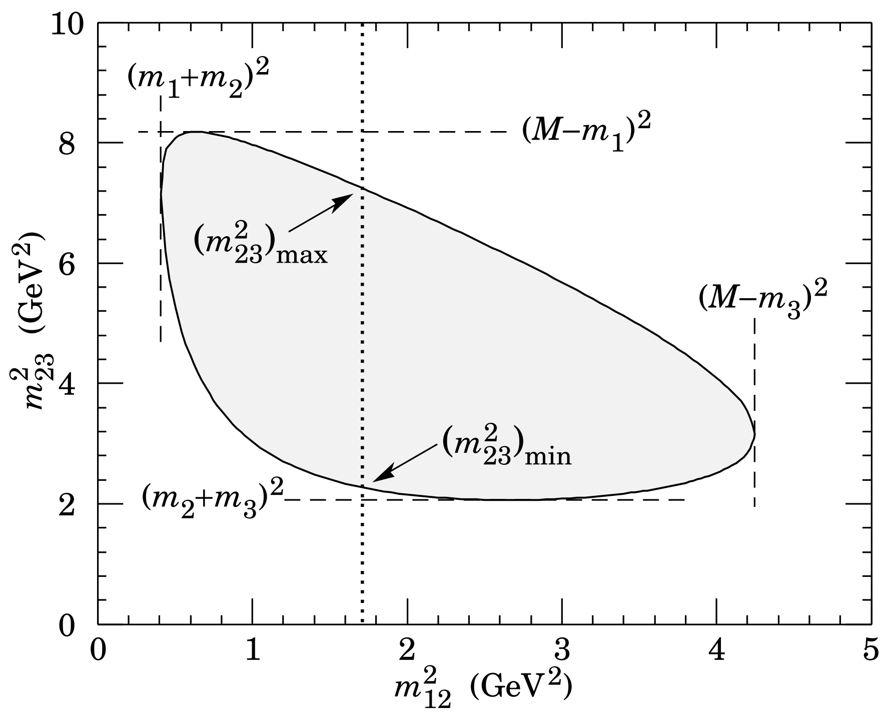

Figure 46.3: Dalitz plot for a three-body final state. In this example, the state is $\pi^+\overline{K}^0 p$ at 3 GeV. Four-momentum conservation restricts events to the shaded region.

46.4.4. *Kinematic limits* :

46.4.4.1. *Three-body decays:* In a three-body decay (Fig. 46.2) the maximum of $|\boldsymbol{p}_3|$, [given by Eq. (46.21)], is achieved when $m_{12} = m_1 + m_2$, *i.e.*, particles 1 and 2 have the same vector velocity in the rest frame of the decaying particle. If, in addition, $m_3 > m_1, m_2$, then $|\boldsymbol{p}_3|_{\rm max} > |\boldsymbol{p}_1|_{\rm max}$, $|\boldsymbol{p}_2|_{\rm max}$. The distribution of m_{12} values possesses an end-point or maximum value at $m_{12} = M - m_3$. This can be used to constrain the mass difference of a parent particle and one invisible decay product.

46.4.4.2. *Sequential two-body decays:*

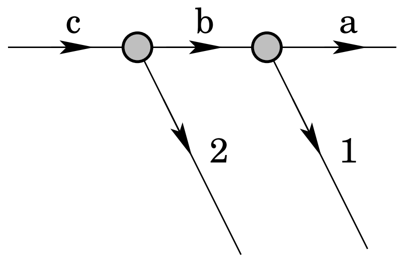

Figure 46.4: Particles participating in sequential two-body decay chain. Particles labeled 1 and 2 are visible while the particle terminating the chain (a) is invisible.

When a heavy particle initiates a sequential chain of two-body decays terminating in an invisible particle, constraints on the masses of the states participating in the chain can be obtained from end-points and thresholds in invariant mass distributions of the aggregated decay products. For the two-step decay chain depicted in Fig. 46.4 the invariant mass distribution of the two visible particles possesses an end-point given by:

$$(m_{12}^{\rm max})^2 = \frac{(m_{\rm c}^2 - m_{\rm b}^2)(m_{\rm b}^2 - m_{\rm a}^2)}{m_{\rm b}^2} \ , \tag{46.24}$$

provided particles 1 and 2 are massless. If visible particle 1 has non-zero mass m_1 then Eq. (46.24) is replaced by

$$(m_{12}^{\rm max})^2 = m_1^2 + \frac{(m_{\rm c}^2 - m_{\rm b}^2)}{2m_{\rm b}^2} \times$$

$$\left(m_1^2 + m_{\rm b}^2 - m_{\rm a}^2 + \sqrt{(-m_1^2 + m_{\rm b}^2 - m_{\rm a}^2)^2 - 4m_1^2 m_{\rm a}^2} \right) . \tag{46.25}$$

See Refs. 2 and 3 for other cases.

46.4.5. ***Multibody decays*** : The above results may be generalized to final states containing any number of particles by combining some of the particles into "effective particles" and treating the final states as 2 or 3 "effective particle" states. Thus, if $p_{ijk\ldots} = p_i + p_j + p_k + \ldots$, then

$$m_{ijk\ldots} = \sqrt{p^2{}_{ijk\ldots}} \; , \tag{46.26}$$

and $m_{ijk\ldots}$ may be used in place of *e.g.*, m_{12} in the relations in Sec. 46.4.3 or Sec. 46.4.4 above.

46.5. Cross sections

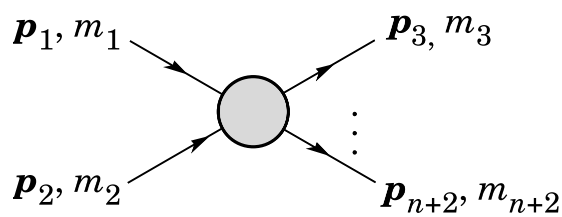

Figure 46.5: Definitions of variables for production of an n-body final state.

The differential cross section is given by

$$d\sigma = \frac{(2\pi)^4 |\mathscr{M}|^2}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \times \; d\Phi_n(p_1 + p_2;\; p_3, \ldots, p_{n+2}) \; . \tag{46.27}$$

[See Eq. (46.12).] In the rest frame of m_2(lab),

$$\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} = m_2 p_{1\,\mathrm{lab}} \; ; \tag{46.28a}$$

while in the center-of-mass frame

$$\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} = p_{1\mathrm{cm}}\sqrt{s} \; . \tag{46.28b}$$

46.5.1. *Two-body reactions* :

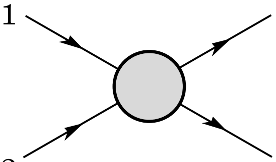

Figure 46.6: Definitions of variables for a two-body final state.

Two particles of momenta p_1 and p_2 and masses m_1 and m_2 scatter to particles of momenta p_3 and p_4 and masses m_3 and m_4; the Lorentz-invariant Mandelstam variables are defined by

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \\ &= m_1^2 + 2E_1E_2 - 2\boldsymbol{p}_1 \cdot \boldsymbol{p}_2 + m_2^2 \ , \end{aligned} \tag{46.29}$$

$$\begin{aligned} t &= (p_1 - p_3)^2 = (p_2 - p_4)^2 \\ &= m_1^2 - 2E_1E_3 + 2\boldsymbol{p}_1 \cdot \boldsymbol{p}_3 + m_3^2 \ , \end{aligned} \tag{46.30}$$

$$\begin{aligned} u &= (p_1 - p_4)^2 = (p_2 - p_3)^2 \\ &= m_1^2 - 2E_1E_4 + 2\boldsymbol{p}_1 \cdot \boldsymbol{p}_4 + m_4^2 \ , \end{aligned} \tag{46.31}$$

and they satisfy

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2 \ . \tag{46.32}$$

The two-body cross section may be written as

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s} \, \frac{1}{|\boldsymbol{p}_{1\text{cm}}|^2} \, |\mathscr{M}|^2 \ . \tag{46.33}$$

In the center-of-mass frame

$$\begin{aligned} t &= (E_{1\text{cm}} - E_{3\text{cm}})^2 - (p_{1\text{cm}} - p_{3\text{cm}})^2 \qquad - 4p_{1\text{cm}} \, p_{3\text{cm}} \, \sin^2(\theta_\text{cm}/2) \\ &= t_0 - 4p_{1\text{cm}} \, p_{3\text{cm}} \, \sin^2(\theta_\text{cm}/2) \ , \end{aligned} \tag{46.34}$$

where θ_cm is the angle between particle 1 and 3. The limiting values t_0 $(\theta_\text{cm} = 0)$ and t_1 $(\theta_\text{cm} = \pi)$ for $2 \to 2$ scattering are

$$t_0(t_1) = \left[\frac{m_1^2 - m_3^2 - m_2^2 + m_4^2}{2\sqrt{s}} \right]^2 - (p_{1\,\text{cm}} \mp p_{3\,\text{cm}})^2 \ . \tag{46.35}$$

In the literature the notation t_min (t_max) for t_0 (t_1) is sometimes used, which should be discouraged since $t_0 > t_1$. The center-of-mass energies and momenta of the incoming particles are

$$E_{1\text{cm}} = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}} \ , \qquad E_{2\text{cm}} = \frac{s + m_2^2 - m_1^2}{2\sqrt{s}} \ , \tag{46.36}$$

For $E_{3\mathrm{cm}}$ and $E_{4\mathrm{cm}}$, change m_1 to m_3 and m_2 to m_4. Then

$$p_{i\,\mathrm{cm}} = \sqrt{E_{i\,\mathrm{cm}}^2 - m_i^2} \text{ and } p_{1\mathrm{cm}} = \frac{p_{1\,\mathrm{lab}}\, m_2}{\sqrt{s}} \ . \tag{46.37}$$

Here the subscript lab refers to the frame where particle 2 is at rest. [For other relations see Eqs. (46.2)–(46.4).]

46.5.2. ***Inclusive reactions*** : Choose some direction (usually the beam direction) for the z-axis; then the energy and momentum of a particle can be written as

$$E = m_T \cosh y \ , \ p_x \ , \ p_y \ , \ p_z = m_T \sinh y \ , \tag{46.38}$$

where m_T, conventionally called the 'transverse mass', is given by

$$m_T^2 = m^2 + p_x^2 + p_y^2 \ . \tag{46.39}$$

and the rapidity y is defined by

$$y = \frac{1}{2} \ln \left(\frac{E + p_z}{E - p_z} \right)$$

$$= \ln \left(\frac{E + p_z}{m_T} \right) = \tanh^{-1} \left(\frac{p_z}{E} \right) \ . \tag{46.40}$$

Note that the definition of the transverse mass in Eq. (46.39) differs from that used by experimentalists at hadron colliders (see Sec. 46.6.1 below). Under a boost in the z-direction to a frame with velocity β, $y \to y - \tanh^{-1} \beta$. Hence the shape of the rapidity distribution dN/dy is invariant, as are differences in rapidity. The invariant cross section may also be rewritten

$$E \frac{d^3\sigma}{d^3p} = \frac{d^3\sigma}{d\phi\, dy\, p_T dp_T} \Longrightarrow \frac{d^2\sigma}{\pi\, dy\, d(p_T^2)} \ . \tag{46.41}$$

The second form is obtained using the identity $dy/dp_z = 1/E$, and the third form represents the average over ϕ.

Feynman's x variable is given by

$$x = \frac{p_z}{p_{z\,\mathrm{max}}} \approx \frac{E + p_z}{(E + p_z)_{\mathrm{max}}} \quad (p_T \ll |p_z|) \ . \tag{46.42}$$

In the c.m. frame,

$$x \approx \frac{2p_{z\,\mathrm{cm}}}{\sqrt{s}} = \frac{2m_T \sinh y_{\mathrm{cm}}}{\sqrt{s}} \tag{46.43}$$

and

$$= (y_{\rm cm})_{\rm max} = \ln(\sqrt{s}/m) \ . \tag{46.44}$$

The invariant mass M of the two-particle system described in Sec. 46.4.2 can be written in terms of these variables as

$$M^2 = m_1^2 + m_2^2 + 2[E_T(1)E_T(2)\cosh\Delta y - \boldsymbol{p}_T(1)\cdot\boldsymbol{p}_T(2)] \ , \tag{46.45}$$

where

$$E_T(i) = \sqrt{|\boldsymbol{p}_T(i)|^2 + m_i^2} \ , \tag{46.46}$$

and $\boldsymbol{p}_T(i)$ denotes the transverse momentum vector of particle i.

For $p \gg m$, the rapidity [Eq. (46.40)] may be expanded to obtain

$$\begin{aligned} y &= \frac{1}{2}\ln\frac{\cos^2(\theta/2) + m^2/4p^2 + \ldots}{\sin^2(\theta/2) + m^2/4p^2 + \ldots} \\ &\approx -\ln\tan(\theta/2) \equiv \eta \end{aligned} \tag{46.47}$$

where $\cos\theta = p_z/p$. The pseudorapidity η defined by the second line is approximately equal to the rapidity y for $p \gg m$ and $\theta \gg 1/\gamma$, and in any case can be measured when the mass and momentum of the particle are unknown. From the definition one can obtain the identities

$$\sinh\eta = \cot\theta \ , \quad \cosh\eta = 1/\sin\theta \ , \quad \tanh\eta = \cos\theta \ . \tag{46.48}$$

46.5.3. ***Partial waves*** : The amplitude in the center of mass for elastic scattering of spinless particles may be expanded in Legendre polynomials

$$f(k,\theta) = \frac{1}{k}\sum_\ell (2\ell+1)a_\ell P_\ell(\cos\theta) \ , \tag{46.49}$$

where k is the c.m. momentum, θ is the c.m. scattering angle, $a_\ell = (\eta_\ell e^{2i\delta_\ell} - 1)/2i$, $0 \le \eta_\ell \le 1$, and δ_ℓ is the phase shift of the ℓ^{th} partial wave. For purely elastic scattering, $\eta_\ell = 1$. The differential cross section is

$$\frac{d\sigma}{d\Omega} = |f(k,\theta)|^2 \ . \tag{46.50}$$

The optical theorem states that

$$\sigma_{\rm tot} = \frac{4\pi}{k}{\rm Im}\, f(k,0) \ , \tag{46.51}$$

and the cross section in the ℓ^{th} partial wave is therefore bounded:

$$\sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)|a_\ell|^2 \le \frac{4\pi(2\ell+1)}{k^2} \ . \tag{46.52}$$

The evolution with energy of a partial-wave amplitude a_ℓ can be displayed as a trajectory in an Argand plot, as shown in Fig. 46.7.

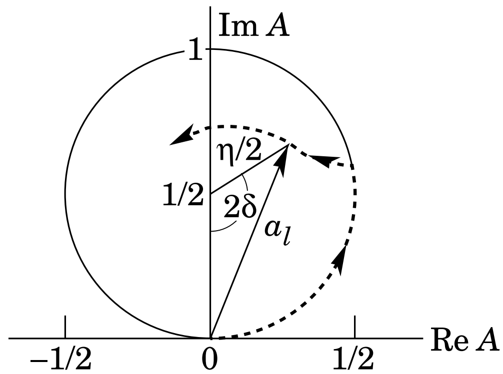

Figure 46.7: Argand plot showing a partial-wave amplitude a_ℓ as a function of energy. The amplitude leaves the unitary circle where inelasticity sets in ($\eta_\ell < 1$).

The usual Lorentz-invariant matrix element $\mathscr{M}$ (see Sec. 46.3 above) for the elastic process is related to $f(k, \theta)$ by

$$\mathscr{M} = -8\pi\sqrt{s}\, f(k, \theta) \ , \tag{46.53}$$

so

$$\sigma_{\rm tot} = -\frac{1}{2p_{\rm lab}\, m_2}\, {\rm Im}\, \mathscr{M}(t = 0) \ , \tag{46.54}$$

where s and t are the center-of-mass energy squared and momentum transfer squared, respectively (see Sec. 46.4.1).

46.5.3.1. *Resonances:* The Breit-Wigner (nonrelativistic) form for an elastic amplitude a_ℓ with a resonance at c.m. energy E_R, elastic width $\Gamma_{\rm el}$, and total width $\Gamma_{\rm tot}$ is

$$a_\ell = \frac{\Gamma_{\rm el}/2}{E_R - E - i\Gamma_{\rm tot}/2} \ , \tag{46.55}$$

where E is the c.m. energy. As shown in Fig. 46.8, in the absence of background the elastic amplitude traces a counterclockwise circle with center $ix_{\rm el}/2$ and radius $x_{\rm el}/2$, where the elasticity $x_{\rm el} = \Gamma_{\rm el}/\Gamma_{\rm tot}$. The amplitude has a pole at $E = E_R - i\Gamma_{\rm tot}/2$.

The spin-averaged Breit-Wigner cross section for a spin-J resonance produced in the collision of particles of spin S_1 and S_2 is

$$\sigma_{BW}(E) = \frac{(2J + 1)}{(2S_1 + 1)(2S_2 + 1)}\, \frac{\pi}{k^2}\, \frac{B_{\rm in} B_{\rm out} \Gamma_{\rm tot}^2}{(E - E_R)^2 + \Gamma_{\rm tot}^2/4} \ , \tag{46.56}$$

where k is the c.m. momentum, E is the c.m. energy, and $B_{\rm in}$ and $B_{\rm out}$ are the branching fractions of the resonance into the entrance and exit channels. The $2S + 1$ factors are the multiplicities of the incident spin states, and are replaced by 2 for photons. This expression is valid only for an isolated state. If the width is not small, $\Gamma_{\rm tot}$ cannot be treated as a constant independent of E. There are many other forms for σ_{BW}, all of which are equivalent to the one given here in the narrow-width case. Some of these forms may be more appropriate if the resonance is broad.

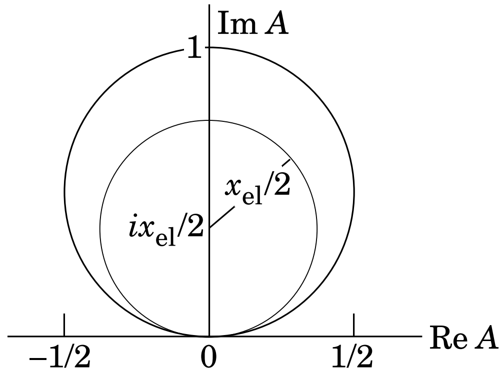

Figure 46.8: Argand plot for a resonance.

The relativistic Breit-Wigner form corresponding to Eq. (46.55) is:

$$a_\ell = \frac{-m\Gamma_{\rm el}}{s - m^2 + im\Gamma_{\rm tot}} \ . \tag{46.57}$$

A better form incorporates the known kinematic dependences, replacing $m\Gamma_{\rm tot}$ by $\sqrt{s}\,\Gamma_{\rm tot}(s)$, where $\Gamma_{\rm tot}(s)$ is the width the resonance particle would have if its mass were $\sqrt{s}$, and correspondingly $m\Gamma_{\rm el}$ by $\sqrt{s}\,\Gamma_{\rm el}(s)$ where $\Gamma_{\rm el}(s)$ is the partial width in the incident channel for a mass $\sqrt{s}$:

$$a_\ell = \frac{-\sqrt{s}\,\Gamma_{\rm el}(s)}{s - m^2 + i\sqrt{s}\,\Gamma_{\rm tot}(s)} \ . \tag{46.58}$$

For the Z boson, all the decays are to particles whose masses are small enough to be ignored, so on dimensional grounds $\Gamma_{\rm tot}(s) = \sqrt{s}\,\Gamma_0/m_Z$, where Γ_0 defines the width of the Z, and $\Gamma_{\rm el}(s)/\Gamma_{\rm tot}(s)$ is constant. A full treatment of the line shape requires consideration of dynamics, not just kinematics. For the Z this is done by calculating the radiative corrections in the Standard Model.

46.6. Transverse variables

At hadron colliders, a significant and unknown proportion of the energy of the incoming hadrons in each event escapes down the beam-pipe. Consequently if invisible particles are created in the final state, their net momentum can only be constrained in the plane transverse to the beam direction. Defining the z-axis as the beam direction, this net momentum is equal to the missing transverse energy vector

$$\boldsymbol{E}_T^{\rm miss} = -\sum_i \boldsymbol{p}_T(i) \ , \tag{46.59}$$

where the sum runs over the transverse momenta of all visible final state particles.

46.6.1. ***Single production with semi-invisible final state*** :

Consider a single heavy particle of mass M produced in association with visible particles which decays as in Fig. 46.1 to two particles, of which one (labeled particle 1) is invisible. The mass of the parent particle can be constrained with the quantity M_T defined by

$$\begin{aligned} M_T^2 &\equiv [E_T(1) + E_T(2)]^2 - [\boldsymbol{p}_T(1) + \boldsymbol{p}_T(2)]^2 \\ &= m_1^2 + m_2^2 + 2[E_T(1)E_T(2) - \boldsymbol{p}_T(1) \cdot \boldsymbol{p}_T(2)] \ , \end{aligned} \tag{46.60}$$

where

$$\boldsymbol{p}_T(1) = \boldsymbol{E}_T^{\rm miss} \ . \tag{46.61}$$

This quantity is called the 'transverse mass' by hadron collider experimentalists but it should be noted that it is quite different from that used in the description of inclusive reactions [Eq. (46.39)]. The distribution of event M_T values possesses an end-point at $M_T^{\rm max} = M$. If $m_1 = m_2 = 0$ then

$$M_T^2 = 2|\boldsymbol{p}_T(1)||\boldsymbol{p}_T(2)|(1 - \cos\phi_{12}) \ , \tag{46.62}$$

where ϕ_{ij} is defined as the angle between particles i and j in the transverse plane.

46.6.2. ***Pair production with semi-invisible final states*** :

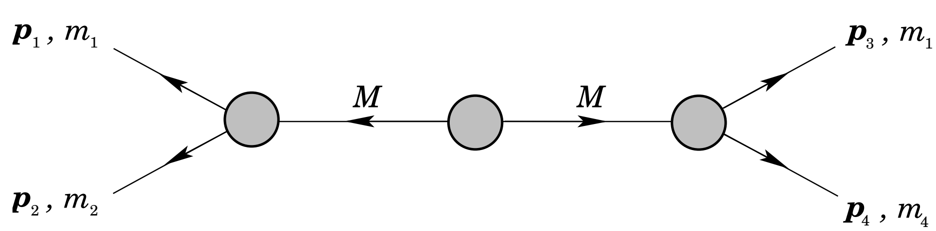

Figure 46.9: Definitions of variables for pair production of semi-invisible final states. Particles 1 and 3 are invisible while particles 2 and 4 are visible.

Consider two identical heavy particles of mass M produced such that their combined center-of-mass is at rest in the transverse plane (Fig. 46.9). Each particle decays to a final state consisting of an invisible particle of fixed mass m_1 together with an additional visible particle. M and m_1 can be constrained with the variables M_{T2} and M_{CT} which are defined in Refs. 4 and 5.

References:

1. See, for example, J.J. Sakurai, *Modern Quantum Mechanics*, Addison-Wesley (1985), p. 172, or D.M. Brink and G.R. Satchler, *Angular Momentum*, 2nd ed., Oxford University Press (1968), p. 20.
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5. D.R. Tovey, JHEP **0804**, 034 (2008).