86. Review of Multibody Charm Analyses

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86.1. Kinematics & Models

The differential decay rate to a point $\mathbf{s} = (s_1, \ldots, s_n)$ in n dimensional phase space can be expressed as

$$d\Gamma = |\mathcal{M}(\mathbf{s})|^2 \left| \frac{\partial^n \phi}{\partial(s_1 \ldots s_n)} \right| d^n s \tag{86.1}$$

where $|\partial^n \phi / \partial(s_1 \ldots s_n)|$ represents the density of states at $\mathbf{s}$, and $\mathcal{M}$ the matrix element for the decay at that point in phase space. For two-body decays, $|\partial^n \phi / \partial(s_1 \ldots s_n)|$ is a δ function, while for D decays to $3, 4, 5, \ldots$ pseudoscalars, phase space is $2, 5, 8, \ldots$ dimensional, leading to a rich phenomenology. Additional parameters are required to fully describe decays with vector particles in the initial or final state.

For the important case of a three body decay $D \to abc$, where D, a, b, c are all pseudoscalars, the decay kinematics can be represented in a two dimensional Dalitz plot [1]. This is usually parametrized in terms of invariant-mass-squared variables $s_1 = (p_a + p_b)^2$ and $s_2 = (p_b + p_c)^2$, where p_a, p_b, p_c are the four-momenta of particles a, b, c. In terms of these variables, phase-space density is constant across the kinematically allowed region, so that any structure seen in the Dalitz plot is a direct consequence of the dynamics encoded in $|\mathcal{M}|^2$. For this type of decay, the operation of parity can also be expressed as a rotation of the decay plane, so no parity violating kinematic observables can be defined (unless they also violate rotational invariance). This is not the case for decays to four or more particles, which can therefore not be unambiguously described in terms of invariant-mass-squared variables, which are parity-even. The use of parity-odd observables in four body decays is discussed below.

The matrix element $\mathcal{M}$ is usually modeled as a sum of interfering decay amplitudes, each proceeding through resonant two-body decays [2]. See Refs [2–4] for a review of resonance phenomenology. In most analyses, each resonance is described by a Breit-Wigner [5] or Flatté [6] lineshape, and the model includes a non-resonant term with a constant phase and magnitude across the Dalitz plot. This approach has well-known theoretical limitations, such as the violation of unitarity and analyticity, which can break the relationship between magnitude and phase across phase space. This motivates the use of more sophisticated descriptions, especially for broad, overlapping resonances (frequently found in S-wave components) where these limitations are particularly problematic. In charm analyses, these approaches have included the K-matrix approach [5,7,8] which respects two-body unitarity; the use of LASS scattering data [9]; dispersive methods [10,11,12,13]; methods based on chiral perturbation theory [14,15] and quasi model-independent parametrizations [16,17,18]. An important example first analyzed by CLEO [19,20,21] is $D^0 \to K_S \pi^+ \pi^-$, which is a key channel in CP violation and charm mixing analyses. Belle models this decay as a superposition of 18 resonances (including 4 significant doubly Cabibbo suppressed amplitudes) described by Breit-Wigner or Flatté lineshapes, plus a non-resonant component [22]. CDF's analysis follows a similar approach [23]. BaBar's model replaces the broad $\pi\pi$ and $K\pi$ S-wave resonances and the non-resonant component with a K-matrix description [24]. Belle's and BaBar's data have been re-analyzed by [25] in a QCD factorization framework, using line-shape parametrizations for the S [26,27] and P wave [11] contributions that preserve 2-body unitarity and analyticity. The measurements give compatible results for the components they share. The field of amplitudes analyses remains very active. Publications since the last update of this review two years ago include Dalitz plot analyses of $D^0 \to K^0_S K^{\pm}\pi^{\mp}$ by LHCb [28], and $D^0 \to \pi^+\pi^-\pi^0$ by BaBar [29]; and several four body amplitude analyses: $D^0 \to K^+K^-\pi^+\pi^-$ and $D^0 \to \pi^+\pi^-\pi^+\pi^-$ using CLEO data [18], and $K^-\pi^+\pi^-\pi^+$ by BES III [30].

All of the examples above remain within the confines of the "isobar" framework, which describes the decay as a series of 2-body amplitudes. While some include very sophisticated descriptions of these 2-body amplitudes, they do not respect the unitarity of the full 3 (or 4) body process and ignore long-range hadronic effects such as rescattering. Several groups work on improved models. Dispersive techniques that respect 3-body unitarity and analyticity by construction have been successfully applied to regions of the $D^+ \to K^-\pi^+\pi^+$ and $D^+ \to K_S\pi^0\pi^+$ Dalitz plots below the $\eta' K$ threshold [12,13], where they provide a good description of the data with fewer fit parameters than the isobar approach. Ref. [31] uses a unitary coupled channel approach to describe $D^+ \to K^-\pi^+\pi^+$, which has no restrictions on the kinematic range, but requires significantly more free parameters. Ref. [14] use chiral perturbation theory to provide a description of the annihilation contribution to the decay amplitude which respects 3-body unitarity.

Limitations in the theoretical description of interfering resonances are the leading source of systematic uncertainty in many analyses. This is set to become increasingly problematic given the statistical precision achievable with the vast, clean charm samples available at the B factories, LHCb, and their upgrades. In some cases, the model uncertainty can be removed through model-independent methods, often relying on input from the charm threshold, as discussed below. At the same time, increasingly sophisticated models are being developed, and applied to data.

86.2. Applications of multibody charm analyses

The interference between the decay paths via which multibody decays proceed provides sensitivity to both relative magnitudes and phases of the contributing amplitudes. It is especially this sensitivity to phases that makes amplitude analyses such a uniquely powerful tool for studying a wide range of phenomena. Here we concentrate on their use for *CP* violation and mixing measurements in charm, and charm inputs to *CP* violation analyses in B meson decays (see also [32,33]) . The properties of light-meson resonances determined in D amplitude analyses are reported in the light-unflavored-meson section of this *Review*.

86.2.1. ***Charm Mixing and CP violation*** : Time-dependent amplitude analyses in decays to final states that are accessible to both D^0 and $\overline{D}^0$ have unique sensitivity to mixing parameters. A Dalitz plot analysis of a self-conjugate final state, such as $K_S\pi^+\pi^-$ and $K_S K^+K^-$, allows the measurement of the phase difference between the relevant D^0 and $\overline{D}^0$ decay amplitudes, and thus a direct measurement of the mixing parameters x, y (rather than the decay-specific parameters x'^2, y' measured for example in $D^0 \to K\pi$) [21]) . These analyses are also sensitive to CP violation in mixing and in the interference between mixing and decay; these results are summarised in [32,33].

86.2.2. ***Measuring γ/ϕ_3*** : Neutral D mesons originating from $B^- \to DK^-$ (here denoted as D_{B^-}) are a superposition of D^0 and $\overline{D}^0$ with a relative phase that depends on γ/ϕ_3:

$$D_{B^-} \propto D^0 + r_B e^{i(\delta_B - \gamma)}\overline{D}^0,$$

where δ_B is a CP conserving strong phase, and $r_B \sim 0.1$. In the corresponding CP-conjugate expression, γ/ϕ_3 changes sign. An amplitude analysis of the subsequent decay of the $D_{B^\pm}$ to a state accessible to both D^0 and $\overline{D}^0$ allows the measurement of γ/ϕ_3 [34–39]. The method generalizes to similar B hadron decays, such as $B^0 \to DK^{*0}$. Measurements based on this technique have been reported by BaBar [40,41], Belle [22,42] and LHCb [43–49,50,51]. The most precise individual results come from the study of $D_{B^-} \to K_S\pi^+\pi^-$ and $D_{B^-} \to K_S K^+K^-$ with an uncertainty of $\sim 15^\circ$ [22,40,42,46]; combining measurements in multiple decay modes leads to a current uncertainty on γ/ϕ_3 of less than 6°.

86.2.3. ***Time-integrated searches for CP violation in charm*** :

Comparing the results of amplitude fits for CP-conjugate decay modes provides a measure of CP violation. Recent CP violation searches using this method include LHCb's amplitude analyses of $D^0 \to K^0_S K^\pm\pi^\mp$ [28], and amplitude analyses of $D^0 \to K^+K^-\pi^+\pi^-$ and $D^0 \to \pi^+\pi^-\pi^+\pi^-$ using CLEO data [52,18].

A widely-used model-independent technique to search for local CP violation is based on performing a χ^2 comparison of CP-conjugate phase-space distributions. This method was pioneered by BaBar [53] and developed further in [54,55,56], with recent results reported by BaBar [57] and LHCb in $D^\pm \to K^+K^-\pi^\pm$ [58,59], by CDF in $D^0 \to K_S\pi^+\pi^-$ [23], and by LHCb in $D^+ \to \pi^-\pi^+\pi^+$ [61], $D^0 \to K^+K^-\pi^+\pi^-$ and $D^0 \to \pi^+\pi^-\pi^+\pi^-$ [56]. Un-binned methods can increase the sensitivity [60] and have been applied by LHCb to $D^+ \to \pi^-\pi^+\pi^+$ [61], $D^0 \to \pi^+\pi^-\pi^0$ [62] and $D^0 \to \pi^+\pi^-\pi^+\pi$ [63].

An alternative model-independent approach, providing complementary information, is based on constructing observables in four body decays that are odd under motion reversal ("naïve T") [64–72], which is equivalent to P for scalar particles [72]. One such observable is $C_T = \vec{p}_2 \cdot (\vec{p}_3 \times \vec{p}_4) = (1/m_D)\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_2^\beta p_3^\gamma p_4^\delta$, where $\vec{p}_i$ are the decay products' three momenta in the decay's restframe, and p_i are their four-momenta. Identical particles (as in $D^0 \to K^+\underline{\pi}^-\pi^+\underline{\pi}^-$) are ordered by momentum magnitude. Comparing the P violating asymmetry $A_T \equiv \dfrac{\Gamma(C_T > 0) - \Gamma(C_T < 0)}{\Gamma(C_T > 0) + \Gamma(C_T < 0)}$ with its C-conjugate in $\overline{D}^0$ decays, provides sensitivity to CP violation.

Searches for CP violation in this manner have been carried out by FOCUS in $D^0 \to K^+K^-\pi^+\pi^-$ [73]; by BaBar in $D^0 \to K^+K^-\pi^+\pi^-$, $D^+ \to K^+K_S\pi^+\pi^-$, and $D_s^+ \to K^+K_S\pi^+\pi^-$ [74,75]; and by LHCb in $D^0 \to K^+K^-\pi^+\pi^-$ [76], where the sensitivity of the method was improved by dividing phase space into bins.

LHCb combined these approaches and use P-odd variables to split their sample of 1M $D^0, \overline{D}^0 \to \pi^+\pi^-\pi^+\pi^-$ events into four sub-samples E, $\exists$, $\overline{E}$, $\overline{\exists}$. Samples E and $\exists$ have opposite C_T and are related by P; the over-line indicates charge conjugation. Comparing $(E + \exists)$ with $(\overline{E} + \overline{\exists})$ tests for P-even CP violation, while comparing $(E + \overline{E})$ with $(\exists + \overline{\exists})$ tests for P-odd CP violation. An unbinned method is used to compare the phase-space distributions of the samples [63].

The results of all measurements described in this section are compatible with CP conservation in charm. In the case of LHCb's P-odd test in $D^0, \overline{D}^0 \to \pi^+\pi^-\pi^+\pi^-$ this compatibility is, with a p-value of 0.6%, marginal.

86.3. Model Independent Methods and the Charm Threshold

The precision measurement of mixing or CP violation parameters (such γ/ϕ_3) from multibody charm decays requires as input both magnitude and phase of the $D^0, \overline{D}^0$ meson decay amplitudes to the final state of interest. While the magnitude is fairly easily measured, the phase information requires either amplitude models with reliable phase motion, or model-independent approaches.

The desired model-independent phase information is accessible at the charm threshold, where CLEO–c and BES III operate [32,37,77–83]. There, D mesons originate from the decay $\psi(3770) \to D\overline{D}$. Quantum-correlations between the two D mesons can be used to identify decays of well-defined $D^0 - \overline{D}^0$ superpositions to the final state of interest. The resulting interference of D^0 and $\overline{D}^0$ decay amplitudes provides observables that depend on the phase difference between those amplitudes, which is the information needed as input for γ/ϕ_3. The measurements are performed either integrated over the entire phase space of the decay, or in sub-regions/bins. The relevant results can be expressed in terms of one complex parameter $\mathcal{Z} = Re^{-i\delta} = c + is$ per pair of CP-conjugate bins, with magnitude $R \leq 1$. Larger R values lead to higher sensitivity to γ/ϕ_3. Amplitude models can be used to optimise the binning for sensitivity to γ/ϕ_3, without introducing a model-dependent bias in the result.

Charm mixing also results in a (time-dependent) superposition of D^0 and $\overline{D}^0$. Charm mixing measurements are therefore sensitive to the same decay-mode specific hadronic phases as γ/ϕ_3 measurements. On one hand, these phases can be seen as nuisance parameters in mixing measurements, which can be constrained at the charm threshold. This is discussed further in Ref. [32]. Conversely, charm mixing can be used to obtain the relevant decay-specific phase information needed for γ/ϕ_3 measurements, using mixing parameters x, y obtained using other charm decay modes as input. This method is particularly powerful in doubly Cabibbo-suppressed decays such as $D^0 \to K^+\pi^-\pi^+\pi^-$, and when used in combination with threshold data [84,85].

CLEO–c data have been analyzed to provide binned $\mathcal{Z}$ for the self-conjugate decays $D^0 \to K_S\pi^+\pi^-$ and $D^0 \to K_SK^+K^-$ [86,87]. For the decay modes

$D^0, \overline{D}^0 \to K_S K^+ \pi^-$, $K^-\pi^+\pi^0$ and $K^-\pi^+\pi^-\pi^+$, phase-space integrated analyses of CLEO–c data have yielded $\mathcal{Z}^{K_S K\pi} = (0.70 \pm 0.08)\exp(-i(0.1^\circ \pm 15.7^\circ))$, $\mathcal{Z}^{K\pi\pi^0} = (0.82 \pm 0.06)\exp(-i(199^\circ{}^{+13^\circ}_{-14^\circ}))$, and $\mathcal{Z}^{K3\pi} = (0.53^{+0.18}_{-0.21})\exp(-i(125^\circ{}^{+22^\circ}_{-14^\circ}))$ [88,89]. Adding input from LHCb's $D^0 \to K^+\pi^-\pi^+\pi^-$ charm mixing measurement changes the latter to $\mathcal{Z}^{K3\pi} = (0.32^{+0.17}_{-0.13})\exp(-i(128^\circ{}^{+28^\circ}_{-17^\circ}))$ [89,90], where the increased uncertainty reported on δ is a consequence of the smaller central value for R. Restricting the analysis of the $K_S K\pi$ final state to a bin around the K^*K resonance, [88] find $R = 0.94 \pm 0.12$, illustrating the benefit in dividing phase space into bins. The above results are given following the usual convention for γ/ϕ_3-related studies where $CP|D^0\rangle = +|\overline{D}^0\rangle$; in the context of charm mixing, it is customary to take $CP|D^0\rangle = -|\overline{D}^0\rangle$, leading to a phase-shift in δ of π.

The corresponding phase space-integrated parameter for self-conjugate decays such as $D^0 \to \pi^+\pi^-\pi^0$ is the real-valued *CP*-even fraction F_+, defined such that a *CP* even eigenstate has $F_+ = 1$, while a *CP*-odd eigenstate has $F_+ = 0$ [81]. A recent analysis of CLEO–c data reveals that $D^0 \to \pi^+\pi^-\pi^0$ is compatible with being completely *CP*-even with $F_+ = 0.973 \pm 0.017$, while $D^0 \to K^+K^-\pi^0$ has $F_+ = 0.732 \pm 0.055$ and $D^0 \to \pi^+\pi^-\pi^+\pi^-$ has $F_+ = 0.737 \pm 0.028$ [82]. Comparing the latter result with the F_+ value derived from the latest $D^0 \to \pi^+\pi^-\pi^+\pi^-$ amplitude model [18], $F_+^{4\pi\ \mathrm{model}} = 0.729 \pm 0.009 \pm 0.018$, provides a useful cross check for the model.

86.4. Summary

Multibody charm decays offer a rich phenomenology, including unique sensitivity to *CP* violation and charm mixing. This is a highly dynamic field with many new results (some of which we presented here) and rapidly increasing, high quality datasets. These datasets constitute a huge opportunity, but also a challenge to improve the theoretical descriptions of soft hadronic effects in multibody decays. For some measurements, model-independent methods, many relying on input from the charm threshold, provide a way of removing model-induced uncertainties. At the same time, work is ongoing to improve the theoretical description of multibody decays.

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