

## 7. Electromagnetic Relations

Revised September 2005 by H.G. Spieler (LBNL).

| Quantity  | Gaussian CGS   | SI  |
|---|--|---|
| Conversion factors:<br>Charge:<br>Potential:<br>Magnetic field: | 2.997 924 58 × 10 <sup>9</sup> esu<br>(1/299.792 458) statvolt (ergs/esu)<br>10 <sup>4</sup> gauss = 10 <sup>4</sup> dyne/esu  | = 1 C = 1 A s<br>= 1 V = 1 J C <sup>-1</sup><br>= 1 T = 1 N A <sup>-1</sup> m <sup>-1</sup>   |
|   | $\mathbf{F} = q(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$  | $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$   |
|   | $\nabla \cdot \mathbf{D} = 4\pi\rho$<br>$\nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} = \frac{4\pi}{c} \mathbf{J}$<br>$\nabla \cdot \mathbf{B} = 0$<br>$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0$                                | $\nabla \cdot \mathbf{D} = \rho$<br>$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}$<br>$\nabla \cdot \mathbf{B} = 0$<br>$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$  |
| Constitutive relations:   | $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ , $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$  | $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$ , $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$   |
| Linear media:   | $\mathbf{D} = \epsilon\mathbf{E}$ , $\mathbf{H} = \mathbf{B}/\mu$<br>1<br>1  | $\mathbf{D} = \epsilon\mathbf{E}$ , $\mathbf{H} = \mathbf{B}/\mu$<br>$\epsilon_0 = 8.854 187 \dots \times 10^{-12}$ F m <sup>-1</sup><br>$\mu_0 = 4\pi \times 10^{-7}$ N A <sup>-2</sup>  |
|   | $\mathbf{E} = -\nabla V - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$<br>$\mathbf{B} = \nabla \times \mathbf{A}$   | $\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$<br>$\mathbf{B} = \nabla \times \mathbf{A}$  |
|   | $V = \sum_{\text{charges}} \frac{q_i}{r_i} = \int \frac{\rho(\mathbf{r}')}{ \mathbf{r} - \mathbf{r}' } d^3x'$<br>$\mathbf{A} = \frac{1}{c} \oint \frac{I d\boldsymbol{\ell}}{ \mathbf{r} - \mathbf{r}' } = \frac{1}{c} \int \frac{\mathbf{J}(\mathbf{r}')}{ \mathbf{r} - \mathbf{r}' } d^3x'$              | $V = \frac{1}{4\pi\epsilon_0} \sum_{\text{charges}} \frac{q_i}{r_i} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{ \mathbf{r} - \mathbf{r}' } d^3x'$<br>$\mathbf{A} = \frac{\mu_0}{4\pi} \oint \frac{I d\boldsymbol{\ell}}{ \mathbf{r} - \mathbf{r}' } = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{ \mathbf{r} - \mathbf{r}' } d^3x'$ |
|   | $\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}$<br>$\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \frac{1}{c} \mathbf{v} \times \mathbf{B})$<br>$\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}$<br>$\mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \frac{1}{c} \mathbf{v} \times \mathbf{E})$ | $\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}$<br>$\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B})$<br>$\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}$<br>$\mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \frac{1}{c^2} \mathbf{v} \times \mathbf{E})$  |
|   | $\frac{1}{4\pi\epsilon_0} = c^2 \times 10^{-7} \text{ N A}^{-2} = 8.987 55 \dots \times 10^9 \text{ m F}^{-1}$ ; $\frac{\mu_0}{4\pi} = 10^{-7} \text{ N A}^{-2}$ ; $c = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 2.997 924 58 \times 10^8 \text{ m s}^{-1}$  |   |

### 7.1. Impedances (SI units)

$\rho$  = resistivity at room temperature in  $10^{-8} \Omega \cdot \text{m}$ :

|                   |                         |
|-------------------|-------------------------|
| $\sim 1.7$ for Cu | $\sim 5.5$ for W        |
| $\sim 2.4$ for Au | $\sim 73$ for SS 304    |
| $\sim 2.8$ for Al | $\sim 100$ for Nichrome |

(Al alloys may have double the Al value.)

For alternating currents, instantaneous current  $I$ , voltage  $V$ , angular frequency  $\omega$ :

$$V = V_0 e^{j\omega t} = ZI. \quad (7.1)$$

Impedance of self-inductance  $L$ :  $Z = j\omega L$ .

Impedance of capacitance  $C$ :  $Z = 1/j\omega C$ .

Impedance of free space:  $Z = \sqrt{\mu_0/\epsilon_0} = 376.7 \Omega$ .

High-frequency surface impedance of a good conductor:

$$Z = \frac{(1+j)\rho}{\delta}, \quad \text{where } \delta = \text{skin depth}; \quad (7.2)$$

$$\delta = \sqrt{\frac{\rho}{\pi\nu\mu}} \approx \frac{6.6 \text{ cm}}{\sqrt{\nu(\text{Hz})}} \quad \text{for Cu}. \quad (7.3)$$

### 7.2. Capacitors, inductors, and transmission Lines

The capacitance between two parallel plates of area  $A$  spaced by the distance  $d$  and enclosing a medium with the dielectric constant  $\epsilon$  is

$$C = K\epsilon A/d, \quad (7.4)$$

where the correction factor  $K$  depends on the extent of the fringing field. If the dielectric fills the capacitor volume without extending beyond the electrodes, the correction factor  $K \approx 0.8$  for capacitors of typical geometry.

The inductance at high frequencies of a straight wire whose length  $\ell$  is much greater than the wire diameter  $d$  is

$$L \approx 2.0 \left[ \frac{\text{nH}}{\text{cm}} \right] \cdot \ell \left( \ln \left( \frac{4\ell}{d} \right) - 1 \right). \quad (7.5)$$

For very short wires, representative of vias in a printed circuit board, the inductance is

$$L(\text{in nH}) \approx \ell/d. \quad (7.6)$$

A transmission line is a pair of conductors with inductance  $L$  and capacitance  $C$ . The characteristic impedance  $Z = \sqrt{L/C}$  and the phase velocity  $v_p = 1/\sqrt{LC} = 1/\sqrt{\mu\epsilon}$ , which decreases with the inverse square root of the dielectric constant of the medium. Typical coaxial and ribbon cables have a propagation delay of about 5 ns/cm.

The impedance of a coaxial cable with outer diameter  $D$  and inner diameter  $d$  is

$$Z = 60 \Omega \cdot \frac{1}{\sqrt{\epsilon_r}} \ln \frac{D}{d}, \quad (7.7)$$

where the relative dielectric constant  $\epsilon_r = \epsilon/\epsilon_0$ . A pair of parallel wires of diameter  $d$  and spacing  $a > 2.5d$  has the impedance

$$Z = 120 \Omega \cdot \frac{1}{\sqrt{\epsilon_r}} \ln \frac{2a}{d}. \quad (7.8)$$

This yields the impedance of a wire at a spacing  $h$  above a ground plane,

$$Z = 60 \Omega \cdot \frac{1}{\sqrt{\epsilon_r}} \ln \frac{4h}{d}. \quad (7.9)$$

A common configuration utilizes a thin rectangular conductor above a ground plane with an intermediate dielectric (microstrip). Detailed calculations for this and other transmission line configurations are given by Gunston.\*

### 7.3. Synchrotron radiation (CGS units)

For a particle of charge  $e$ , velocity  $v = \beta c$ , and energy  $E = \gamma mc^2$ , traveling in a circular orbit of radius  $R$ , the classical energy loss per revolution  $\delta E$  is

$$\delta E = \frac{4\pi}{3} \frac{e^2}{R} \beta^3 \gamma^4. \quad (7.10)$$

For high-energy electrons or positrons ( $\beta \approx 1$ ), this becomes

$$\delta E (\text{in MeV}) \approx 0.0885 [E(\text{in GeV})]^4 / R(\text{in m}). \quad (7.11)$$

For  $\gamma \gg 1$ , the energy radiated per revolution into the photon energy interval  $d(\hbar\omega)$  is

$$dI = \frac{8\pi}{9} \alpha \gamma F(\omega/\omega_c) d(\hbar\omega), \quad (7.12)$$

where  $\alpha = e^2/\hbar c$  is the fine-structure constant and

$$\omega_c = \frac{3\gamma^3 c}{2R} \quad (7.13)$$

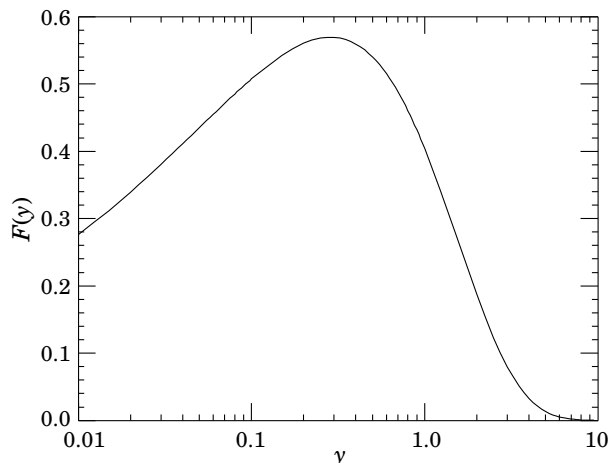
is the critical frequency. The normalized function  $F(y)$  is

$$F(y) = \frac{9}{8\pi} \sqrt{3} y \int_y^\infty K_{5/3}(x) dx, \quad (7.14)$$

where  $K_{5/3}(x)$  is a modified Bessel function of the third kind. For electrons or positrons,

$$\hbar\omega_c (\text{in keV}) \approx 2.22 [E(\text{in GeV})]^3 / R(\text{in m}). \quad (7.15)$$

Fig. 7.1 shows  $F(y)$  over the important range of  $y$ .



**Figure 7.1:** The normalized synchrotron radiation spectrum  $F(y)$ .

For  $\gamma \gg 1$  and  $\omega \ll \omega_c$ ,

$$\frac{dI}{d(\hbar\omega)} \approx 3.3\alpha (\omega R/c)^{1/3}, \quad (7.16)$$

whereas for

$$\gamma \gg 1 \text{ and } \omega \gtrsim 3\omega_c,$$

$$\frac{dI}{d(\hbar\omega)} \approx \sqrt{\frac{3\pi}{2}} \alpha \gamma \left( \frac{\omega}{\omega_c} \right)^{1/2} e^{-\omega/\omega_c} \left[ 1 + \frac{55}{72} \frac{\omega_c}{\omega} + \dots \right]. \quad (7.17)$$

The radiation is confined to angles  $\lesssim 1/\gamma$  relative to the instantaneous direction of motion. For  $\gamma \gg 1$ , where Eq. (7.12) applies, the mean number of photons emitted per revolution is

$$N_\gamma = \frac{5\pi}{\sqrt{3}} \alpha \gamma, \quad (7.18)$$

and the mean energy per photon is

$$\langle \hbar\omega \rangle = \frac{8}{15\sqrt{3}} \hbar\omega_c. \quad (7.19)$$

When  $\langle \hbar\omega \rangle \gtrsim O(E)$ , quantum corrections are important.

\* M.A.R. Gunston. Microwave Transmission Line Data, Noble Publishing Corp., Atlanta (1997) ISBN 1-884932-57-6, TK6565.T73G85.

See J.D. Jackson, *Classical Electrodynamics*, 3<sup>rd</sup> edition (John Wiley & Sons, New York, 1998) for more formulae and details. (Note that earlier editions had  $\omega_c$  twice as large as Eq. (7.13).)