74. Rare Kaon Decays

Revised September 2017 by L. Littenberg (BNL) and G. Valencia (Monash University).

74.1. Introduction

There are several useful reviews on rare kaon decays and related topics [1–17]. Activity in rare kaon decays can be divided roughly into four categories:

1. Searches for explicit violations of the Standard Model (SM)
2. The golden modes: $K \to \pi\nu\bar{\nu}$
3. Other constraints on SM parameters
4. Studies of strong interactions at low energy.

The paradigm of Category 1 is the lepton flavor violating decay $K_L \to \mu e$. Category 2 includes the two modes that can be calculated with negligible theoretical uncertainty, $K^+ \to \pi^+\nu\overline{\nu}$ and $K_L \to \pi^0\nu\overline{\nu}$. These modes can lead to precision determinations of CKM parameters or, in combination with other measurements of these parameters, they can constrain new interactions. They constitute the main focus of the current experimental kaon program. Category 3 is focused on decays with charged leptons, such as $K_L \to \pi^0\ell^+\ell^-$ or $K_L \to \ell^+\ell^-$ where $\ell \equiv e, \mu$. These modes are sensitive to CKM parameters but they suffer from multiple hadronic uncertainties that can be addressed, at least in part, through a systematic study of the peripheral modes indicated in Fig. 74.1. The interplay between Categories 3-4 and their complementarity to Category 2 is illustrated in the figure. Category 4 includes reactions like $K^+ \to \pi^+\ell^+\ell^-$ where long distance contributions are dominant and which constitute a testing ground for the ideas of chiral perturbation theory. Other decays in this category are $K_L \to \pi^0\gamma\gamma$ and $K_L \to \ell^+\ell^-\gamma$. The former is important in understanding a CP-conserving contribution to $K_L \to \pi^0\ell^+\ell^-$, whereas the latter could shed light on long distance contributions to $K_L \to \mu^+\mu^-$.

74.2. Explicit violations of the Standard Model

Much activity has focussed on searches for lepton flavor violation (LFV). This is motivated by the fact that many extensions of the minimal Standard Model violate lepton flavor and by the potential to access very high energy scales. For example, the tree-level exchange of a LFV vector boson of mass M_X that couples to left-handed fermions with electroweak strength and without mixing angles yields $\mathrm{B}(K_L \to \mu e) = 4.7 \times 10^{-12}(148\ \mathrm{TeV}/M_X)^4$ [4]. This simple dimensional analysis may be used to read from Table 74.1 that the reaction $K_L \to \mu e$ is already probing scales of over 100 TeV. Table 74.1 summarizes the present experimental situation vis-à-vis LFV. The decays $K_L \to \mu^\pm e^\mp$ and $K^+ \to \pi^+ e^\mp \mu^\pm$ (or $K_L \to \pi^0 e^\mp \mu^\pm$) provide complementary information on potential family number violating interactions, since the former is sensitive to parity-odd couplings and the latter is sensitive to parity-even couplings.

Limits on certain lepton-number violating (LNV) kaon decays also have been obtained, with recent interest arising from their role in constraining possible extensions of the neutrino sector [18], and we list those in the table as well. Related searches in μ and τ processes are discussed in our section "Tests of Conservation Laws."

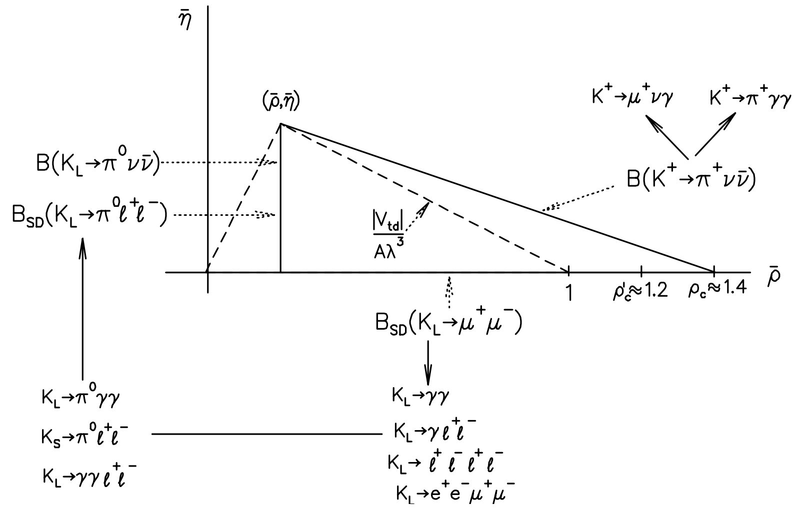

Figure 74.1: Role of rare kaon decays in determining the unitarity triangle. The solid arrows point to auxiliary modes needed to interpret the main results, or potential backgrounds to them.

Table 74.1: Searches for lepton flavor and lepton number violation in K decay

LFV Mode	90% CL upper limit	Exp't	Yr./Ref.	Type
$K^+ \to \pi^+ e^- \mu^+$	1.3×10^{-11}	BNL-865	2005/Ref. 19	LFV
$K^+ \to \pi^+ e^+ \mu^-$	5.2×10^{-10}	BNL-865	2000/Ref. 20	LFV
$K_L \to \mu e$	4.7×10^{-12}	BNL-871	1998/Ref. 21	LFV
$K_L \to \pi^0 e\mu$	7.6×10^{-11}	KTeV	2008/Ref. 22	LFV
$K_L \to \pi^0 \pi^0 e\mu$	1.7×10^{-10}	KTeV	2008/Ref. 22	LFV
$K^+ \to \pi^- e^+ e^+$	6.4×10^{-10}	BNL-865	2000/Ref. 20	LNV
$K^\pm \to \pi^\mp \mu^\pm \mu^\pm$	8.6×10^{-11}	NA48/2	2017/Ref. 23	LNV
$K_L \to e^\pm e^\pm \mu^\mp \mu^\mp$	4.12×10^{-11}	KTeV	2003/Ref. 24	LNV
$K^+ \to \pi^- \mu^+ e^+$	5.0×10^{-10}	BNL-865	2000/Ref. 20	LNFV

Physics beyond the SM is also pursued through the search for $K^+ \to \pi^+ X^0$, where X^0 is a new light particle. The searches cover both long-lived particles (*e.g.*, hyperphoton, axion, familon, *etc.*), and short-lived ones that decay to muon, electron or photon pairs. The 90% CL upper limit on $K^+ \to \pi^+ X^0$ is 7.3×10^{-11} [25] for the case of massless X^0; additional results as a function of the X^0 mass can be found in [26]. Recently these limits have been reinterpreted in connection with a dark photon [27] or dark

Z [28]. Such vectors have also been sought in their e^+e^- decay mode by NA48/2 [29]. Additional bounds for a short lived pseudoscalar X^0 decaying to muons or photons are $B(K_L \to \pi^0\pi^0\mu^+\mu^-) < 1 \times 10^{-10}$ [30] and $B(K_L \to \pi^0\pi^0\gamma\gamma) < 2.4 \times 10^{-7}$ [31].

74.3. The golden modes: $K \to \pi\nu\bar{\nu}$

In the SM, the decay $K^+ \to \pi^+\nu\overline{\nu}$ is dominated by one-loop diagrams with top-quark intermediate states while long-distance contributions are known to be quite small [2,32,33]. This permits a precise calculation of this rate in terms of SM parameters. Studies of this process are thus motivated by the possibility of detecting non-SM physics when comparing with the results of global fits [34,35].

The branching ratio can be written in a compact form that exhibits the different ingredients that go into the calculation [36],

$$\mathrm{B}(K^+ \to \pi^+\nu\overline{\nu}(\gamma)) = \kappa_+(1+\Delta_{\mathrm{EM}})\left[\left(\frac{\mathrm{Im}(V_{ts}^*V_{td})}{\lambda^5}X_t\right)^2 + \left(\frac{\mathrm{Re}(V_{cs}^*V_{cd})}{\lambda}(P_c+\delta P_{c,u}) + \frac{\mathrm{Re}(V_{ts}^*V_{td})}{\lambda^5}X_t\right)^2\right]. \quad (74.1)$$

The parameters in Eq. (74.1) incorporate the *a priori* unknown hadronic matrix element in terms of the very well-measured K_{e3} rate [2] in κ_+; long distance QED corrections in Δ_{EM} [37]; the Inami-Lim function for the short distance top-quark contribution [38] including NLO QCD corrections [39] and the two-loop electroweak correction [36], all in X_t; and the charm-quark contributions due to short distance effects including NNLO QCD corrections [40] and NLO electroweak corrections via P_c [41], as well as certain long distance effects via $\delta P_{c,u}$ [42,33]. An interesting approximate way to cast this result in terms of the CKM parameters λ, V_{cb}, $\overline{\rho}$ and $\overline{\eta}$ (see our Section on "The Cabibbo-Kobayashi-Maskawa mixing matrix") [11] is:

$$\mathrm{B}(K^+ \to \pi^+\nu\overline{\nu}) \approx 1.6 \times 10^{-5}|V_{cb}|^4[\sigma\overline{\eta}^2 + (\rho_c - \overline{\rho})^2], \quad (74.2)$$

where $\rho_c \approx 1.45$ and $\sigma \equiv 1/(1-\frac{1}{2}\lambda^2)^2$. Thus, $\mathrm{B}(K^+ \to \pi^+\nu\overline{\nu})$ determines an ellipse in the $\overline{\rho}$, $\overline{\eta}$ plane with center $(\rho_c, 0)$ and semiaxes $\approx \frac{1}{|V_{cb}|^2}\sqrt{\frac{\mathrm{B}(K^+\to\pi^+\nu\overline{\nu})}{1.6\times10^{-5}}}$ and $\frac{1}{\sigma|V_{cb}|^2}\sqrt{\frac{\mathrm{B}(K^+\to\pi^+\nu\overline{\nu})}{1.6\times10^{-5}}}$.

BNL-787 observed two candidate events [43,44] in the clean high π^+ momentum and one event [45] in the low-momentum region. The successor experiment BNL-949 observed one more in the high-momentum region [25] and three more in the low-momentum region [46], yielding a branching ratio of $(1.73^{+1.15}_{-1.05}) \times 10^{-10}$ [26]. The NA62 experiment [47], performed with in-flight decays at CERN, aims to reach a sensitivity of $\sim 10^{-12}$/event. NA62 was commissioned in 2015 and is expected to reach

SM sensitivities with the data taken in 2016. The 2017 run, presently in progress, is anticipated to produce more than 10 SM events, and the collaboration expects that the experiment will achieve its full sensitivity by the end of the 2018 run.

Our estimate for this branching ratio, using the latest CKMfitter input [34], is $B(K^+ \to \pi^+\nu\overline{\nu}) = (8.3 \pm 0.4) \times 10^{-11}$, near the lower end of the measurement of BNL-787 and 949. However, current parametric uncertainty in the CKM angles can result in numbers with central values differing from this one by up to 10% [48].

The second golden mode is the neutral counterpart to our preceeding discussion: $K_L \to \pi^0\nu\overline{\nu}$. It is dominantly CP-violating and free of hadronic uncertainties [2,49,50]. In the Standard Model, this mode is dominated by an intermediate top-quark state and does not suffer from the small uncertainty associated with the charm-quark intermediate state that affects $K^+ \to \pi^+\nu\overline{\nu}$. The branching ratio is given by Ref. 11:

$$\begin{aligned} B(K_L \to \pi^0\nu\overline{\nu}) &= \kappa_L \left(\frac{\mathrm{Im}(V_{ts}^* V_{td})}{\lambda^5} X_t \right)^2 \\ &\approx 7.6 \times 10^{-5} |V_{cb}|^4 \overline{\eta}^2 \ . \end{aligned} \tag{74.3}$$

As with the charged mode, the hadronic matrix element can be related to that measured in $K_{\ell 3}$ decay and is parameterized in κ_L.

Our estimate for the branching ratio, using the latest CKMfitter input [34], is $(2.9 \pm 0.2) \times 10^{-11}$. But similarly to the charged kaon case, parametric uncertainty in the CKM angles can result in a central value that differs from this one by up to almost 20% [48].

Grossman and Nir (GN) [51] pointed out that, in a nearly model-independent manner, the two golden modes satisfy the relation $B(K_L \to \pi^0\nu\overline{\nu}) \lesssim 4.4\ B(K^+ \to \pi^+\nu\overline{\nu})$. Using the 90% CL bound on $K^+ \to \pi^+\nu\overline{\nu}$, GN then predict $B(K_L \to \pi^0\nu\overline{\nu}) < 1.46 \times 10^{-9}$.

KEK-391a, which took data in 2004 and 2005, has published a 90% CL upper bound of $B(K_L \to \pi^0\nu\overline{\nu}) \leq 2.6 \times 10^{-8}$ [52]. The KOTO experiment at J-PARC [53], whose initial goal is to observe this decay, had a short physics run in the spring of 2013, obtaining a 90% CL upper limit of 5.1×10^{-8} [54]. They resumed running in 2015 and have continued to do so each year, making incremental upgrades to the experimental configuration between runs. They expect to reach the GN bound level from the combined 2015 and 2016 data. It was pointed out in a recent paper that the GN bound quoted above applies to the three body decay $K_L \to \pi^0\nu\bar{\nu}$ and not necessarily to two body modes such as $K_L \to \pi^0 X^0$. In this case KOTO can provide interesting constraints on new physics even at the current sensitivity level [55]. Using the 2013 run, they have established a 90% CL upper limit of 3.7×10^{-8} on $K_L \to \pi^0 X^0$ for $m_{X^0} \approx m_{\pi^0}$ [54].

The current theoretical and experimental situation for the golden modes is summarized in Fig. 74.2. The red area corresponds to the 90% CL SM prediction we obtain with the latest input available from CKMfitter [34]. The dashed yellow region shows the 90% CL region established by the combined BNL-787 and BNL-949 results. The black dashed region illustrates the GN exclusion, which lies significantly above the SM expectation leaving a large window for discovery of new physics contributions by experiments seeking to measure $\mathrm{B}(K_L \to \pi^0 \nu\overline{\nu})$. Much theoretical work has explored beyond the SM scenarios that can populate this window as well as their correlations with other rare processes outside kaon physics. Although it would be relatively straight forward to establish the existence of new physics by observing deviations from their SM values in the $K \to \pi\nu\bar{\nu}$ modes, it would take much more extensive global fits to pinpoint the origin of any such deviation. Partial summaries with references can be found in Refs. [14,56,57,58,59,60].

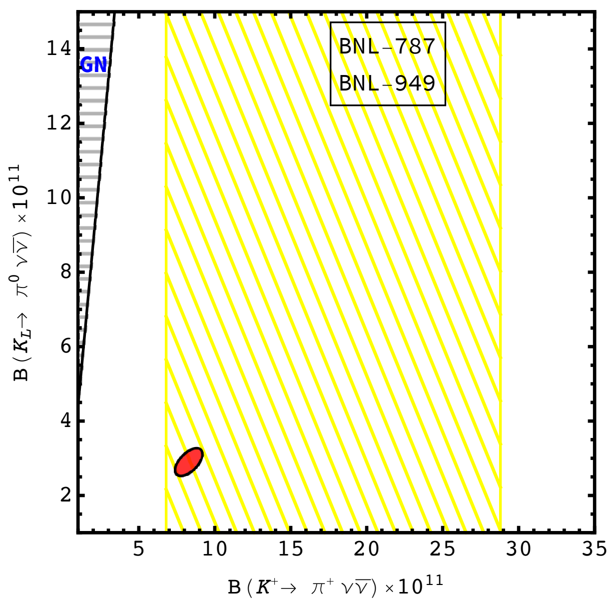

Figure 74.2: Summary of current situation for the golden modes $K \to \pi\nu\bar{\nu}$. The red and yellow regions correspond to the 90% CL SM prediction with input from CKMfitter and to the BNL measurement respectively. The black dashed region shows the GN exclusion.

Related modes with one extra pion, $K \to \pi\pi\nu\bar{\nu}$, are are similarly dominated by short distance contributions [61]. However, they occur at much lower rates with branching rations of order 10^{-13}. The current best bound comes from KEK-391a, it is $B(K_L \to \pi^0\pi^0\nu\bar{\nu}) < 8.1 \times 10^{-7}$ at 90% CL [62]. There is also a bound $B(K^+ \to \pi^+\pi^0\nu\bar{\nu}) < 4.3 \times 10^{-5}$ at 90% CL [63] from BNL-787.

74.4. Other constraints on Standard Model parameters

The decay $K_L \to \mu^+\mu^-$ has a short distance contribution sensitive to the CKM parameter $\overline{\rho}$, given by [11]:

$$\mathrm{B_{SD}}(K_L \to \mu^+\mu^-) \approx 2.7 \times 10^{-4}|V_{cb}|^4(\rho_c' - \overline{\rho})^2 \tag{74.4}$$

where ρ_c' depends on the charm quark mass and is approximately 1.2. This decay, however, is dominated by a long-distance contribution from a two-photon intermediate state. The absorptive (imaginary) part of the long-distance component is determined by the measured rate for $K_L \to \gamma\gamma$ to be $\mathrm{B_{abs}}(K_L \to \mu^+\mu^-) = (6.64 \pm 0.07) \times 10^{-9}$; and it almost completely saturates the observed rate $\mathrm{B}(K_L \to \mu^+\mu^-) = (6.84 \pm 0.11) \times 10^{-9}$ [64]. The difference between the observed rate and the absorptive component can be attributed to the (coherent) sum of the short-distance amplitude and the real part of the long-distance amplitude. The latter cannot be derived directly from experiment [65], but can be estimated with certain assumptions [66,67].

By contrast, the decay $K_L \to e^+e^-$ is completely dominated by long distance physics and is easier to estimate. The result, $B(K_L \to e^+e^-) \sim 9 \times 10^{-12}$ [65,68], is in good agreement with the BNL-871 measurement, $(8.7^{+5.7}_{-4.1}) \times 10^{-12}$ [69].

The mode $K_S \to \mu^+\mu^-$ similarly has a short distance contribution proportional to the square of the CKM parameter $\bar{\eta}$ entering at the 10^{-13} level [15]. It has as well long distance contributions which arising from the two photon intermediate state which result in a rate $B(K_S \to \mu^+\mu^-)_{LD} = 5.1 \times 10^{-12}$ [15]. A 95% (90%) CL limit $B(K_S \to \mu^+\mu^-) < 0.8(1.0) \times 10^{-9}$ was recently obtained by LHCb [70].

The decay $K_L \to \pi^0 e^+e^-$ is sensitive to the CKM parameter η through its CP-violating component. There are both direct and indirect CP-violating amplitudes that can interfere. The direct CP-violating amplitude is short distance dominated and has been calculated in detail within the SM [8]. The indirect CP-violating amplitude can be inferred from a measurement of $K_S \to \pi^0 e^+e^-$. The complete CP-violating contribution to the rate can be written as [71,72]:

$$\mathrm{B_{CPV}} \approx 10^{-12}\left[15.7|a_S|^2 \pm 1.4\left(\frac{|V_{cb}|^2\overline{\eta}}{10^{-4}}\right)|a_S| + 0.12\left(\frac{|V_{cb}|^2\overline{\eta}}{10^{-4}}\right)^2\right] \tag{74.5}$$

where the three terms correspond to the indirect CP violation, the interference, and the direct CP violation, respectively. The parameter a_S has been extracted by NA48/1 from a measurement of $K_S \to \pi^0 e^+e^-$ with the result $|a_S| = 1.06^{+0.26}_{-0.21} \pm 0.07$ [73], as well as from a measurement of $K_S \to \pi^0\mu^+\mu^-$ with the result $|a_s| = 1.54^{+0.40}_{-0.32} \pm 0.06$ [74]. With current constraints on the CKM parameters, and assuming a positive sign for the interference term [72,75], this implies that $\mathrm{B_{CPV}}(K_L \to \pi^0 e^+e^-) \approx (3.1 \pm 0.9) \times 10^{-11}$, where the three contributions to the central value from indirect, interference and direct CP violation are $(1.76, 0.9, 0.45) \times 10^{-11}$ respectively. The complete CP violating amplitude for the related mode $K_L \to \pi^0\mu^+\mu^-$ is predicted to be $\mathrm{B_{CPV}}(K_L \to \pi^0\mu^+\mu^-) \approx (1.4 \pm 0.5) \times 10^{-11}$ [76,15].

$K_L \to \pi^0 e^+e^-$ also has a CP-conserving component dominated by a two-photon intermediate state. This component can be decomposed into an absorptive and a dispersive part. The absorptive part can be extracted from the measurement of the low $m_{\gamma\gamma}$ region of the $K_L \to \pi^0\gamma\gamma$ spectrum. The rate and the shape of the distribution $d\Gamma/dm_{\gamma\gamma}$ in $K_L \to \pi^0\gamma\gamma$ are well described in chiral perturbation theory in terms of three (*a priori*) unknown parameters [77,78].

Both KTeV and NA48 have studied the mode $K_L \to \pi^0\gamma\gamma$, reporting similar results. KTeV finds $B(K_L \to \pi^0\gamma\gamma) = (1.29 \pm 0.03_{\mathrm{stat}} \pm 0.05_{\mathrm{sys}}) \times 10^{-6}$ [79], while NA48 finds $B(K_L \to \pi^0\gamma\gamma) = (1.36 \pm 0.03_{\mathrm{stat}} \pm 0.03_{\mathrm{sys}} \pm 0.03_{\mathrm{norm}}) \times 10^{-6}$ [80]. Both experiments are consistent with a negligible rate in the low $m_{\gamma\gamma}$ region, suggesting a very small CP-conserving component $B_{\mathrm{CP}}(K_L \to \pi^0 e^+e^-) \sim \mathcal{O}(10^{-13})$ [72,78,80]. There remains some model dependence in the estimate of the dispersive part of the CP-conserving $K_L \to \pi^0 e^+e^-$ [72].

The related process, $K_L \to \pi^0\gamma e^+e^-$, is potentially an additional background to $K_L \to \pi^0 e^+e^-$ in some region of phase space [81]. This process has been observed with a branching ratio of $(1.62 \pm 0.14_{\mathrm{stat}} \pm 0.09_{\mathrm{sys}}) \times 10^{-8}$ [82].

The decay $K_L \to \gamma\gamma e^+e^-$ constitutes the dominant background to $K_L \to \pi^0 e^+e^-$. It was first observed by BNL-845 [83], and subsequently confirmed with a much larger sample by KTeV [84]. It has been estimated that this background will enter at about the 10^{-10} level [85,86], comparable to or larger than the signal level. Because of this, the observation of $K_L \to \pi^0 e^+e^-$ at the SM level will depend on background subtraction with good statistics. Possible alternative strategies are discussed in Ref. 72 and references cited therein.

The 90% CL upper bound for the process $K_L \to \pi^0 e^+e^-$ is 2.8×10^{-10} [86]. For the closely related muonic process, the published upper bound is $\mathrm{B}(K_L \to \pi^0\mu^+\mu^-) \leq 3.8 \times 10^{-10}$ [87], compared with the SM prediction of $(1.5 \pm 0.3) \times 10^{-11}$ [76] (assuming positive interference between the direct- and indirect-CP violating components).

A study of $K_L \to \pi^0\mu^+\mu^-$ has indicated that it might be possible to extract the direct CP-violating contribution by a joint study of the Dalitz plot variables and the components of the μ^+ polarization [88]. The latter tends to be quite substantial so that large statistics may not be necessary.

Combined information from $K_L \to \pi^0\ell^+\ell^-$ as well as $K_L \to \mu^+\mu^-$ complements the $K \to \pi\nu\overline{\nu}$ measurements in constraining physics beyond the SM [89].

74.5. Other long distance dominated modes

The decays $K^+ \to \pi^+\ell^+\ell^-$ ($\ell = e$ or μ) have received considerable attention. The rate and spectrum have been measured for both the electron and muon modes [90,91].

The measurements have been used to exclude new physics such as a dark photon [27]. Ref. 71 has proposed a parameterization inspired by chiral perturbation theory, which provides a successful description of data but indicates the presence of large corrections beyond leading order. More work is needed to fully understand the origin of these large corrections. The mode $K^+ \to \pi^+\pi^0 e^+e^-$, recently analyzed by NA48/2 [92], is also dominated by long distance physics but it has been argued that measuring asymmetries can provide information on the short distance components [93]. The related mode $K_S \to \pi^+\pi^- e^+e^-$, which was measured by NA48/1 [94], has received new interest by LHCb [95] as an important background to other rare decays.

The decay $K^+ \to \pi^+\gamma\gamma$ can be predicted in terms of one unknown parameter to leading order in χPT resulting in a correlation between the rate and the diphoton mass spectrum [96]. Certain important corrections at the next order are also known [97]. The rate was first measured by E787 [98], and more recently NA48/2 [99] has obtained a more precise result with a 6% error, as well as the corresponding spectrum fits. The most recent, and precise, result is from NA62 based on a sample of 232 events [100] but is still insufficient to distinguish between the leading order and next order χPT parameterizations.

Much information has been recorded by KTeV and NA48 on the rates and spectrum for the Dalitz pair conversion modes $K_L \to \ell^+\ell^-\gamma$ [101,102], and $K_L \to \ell^+\ell^-\ell'^+\ell'^-$ for $\ell, \ell' = e$ or μ [24,103]. More recently, LHCb has performed preliminary studies of $K_S \to \ell^+\ell^-\ell'^+\ell'^-$ [95]. All these results are used to test hadronic models and should eventually help unravel the underlying physics in $K_L \to \mu^+\mu^-$ [67,104,105].

References:

1. D. Bryman, Int. J. Mod. Phys. **A4**, 79 (1989).
2. J. Hagelin and L. Littenberg, Prog. in Part. Nucl. Phys. **23**, 1 (1989).
3. L. Littenberg and G. Valencia, Ann. Rev. Nucl. and Part. Sci. **43**, 729 (1993).
4. J. Ritchie and S. Wojcicki, Rev. Mod. Phys. **65**, 1149 (1993).
5. B. Winstein and L. Wolfenstein, Rev. Mod. Phys. **65**, 1113 (1993).
6. G. D'Ambrosio *et al.*, *Radiative Non-Leptonic Kaon Decays*, in The DAΦNE Physics Handbook (second edition), eds. L. Maiani, G. Pancheri, and N. Paver (Frascati), Vol. I, 265 (1995).
7. A. Pich, Rept. on Prog. in Phys. **58**, 563 (1995).
8. G. Buchalla, A.J. Buras, and M.E. Lautenbacher, Rev. Mod. Phys. **68**, 1125 (1996).
9. G. D'Ambrosio and G. Isidori, Int. J. Mod. Phys. **A13**, 1 (1996).
10. P. Buchholz and B. Renk Prog. in Part. Nucl. Phys. **39**, 253 (1997).
11. A.J. Buras and R. Fleischer, TUM-HEP-275-97, `hep-ph/9704376`, *Heavy Flavours II*, World Scientific, eds. A.J. Buras and M. Lindner (1997), 65–238.
12. A.J. Buras, TUM-HEP-349-99, Lectures at Lake Louise Winter Institute: Electroweak Physics, Lake Louise, Alberta, Canada, 14–20 Feb. 1999.

13. A.R. Barker and S.H. Kettell, Ann. Rev. Nucl. and Part. Sci. **50**, 249 (2000).
14. A.J. Buras, F. Schwab, and S. Uhlig, Rev. Mod. Phys. **80**, 965 (2008).
15. V. Cirigliano *et al.*, Rev. Mod. Phys. **84**, 399 (2012).
16. D. Bryman *et al.*, Ann. Rev. Nucl. and Part. Sci. **61**, 331 (2011).
17. T.K. Komatsubara, Prog. in Part. Nucl. Phys. **67**, 995 (2012).
18. A. Atre *et al.*, JHEP **0905**, 030 (2009);
    L.S. Littenberg and R.E. Shrock, Phys. Lett. **B491**, 285 (2000).
19. A. Sher *et al.*, Phys. Rev. **D72**, 012005 (2005).
20. R. Appel *et al.*, Phys. Rev. Lett. **85**, 2877 (2000).
21. D. Ambrose *et al.*, Phys. Rev. Lett. **81**, 5734 (1998).
22. E. Abouzaid *et al.*, Phys. Rev. Lett. **100**, 131803 (2008).
23. J.R. Batley *et al.*, Phys. Lett. **B769**, 67 (2017).
24. A. Alavi-Harati *et al.*, Phys. Rev. Lett. **90**, 141801 (2003).
25. V.V. Anisimovsky *et al.*Phys. Rev. Lett. **93**, 031801 (2004).
26. A.V. Artamonov *et al.*, Phys. Rev. **D79**, 092004 (2009).
27. M. Pospelov, Phys. Rev. **D80**, 095002 (2009).
28. H. Davoudiasl, H.S. Lee, and W.J. Marciano, Phys. Rev. **D89**, 095006 (2014).
29. J.R. Batley *et al.* [NA48/2 Collab.], Phys. Lett. **B746**, 178 (2015).
30. E. Abouzaid *et al.*, Phys. Rev. Lett. **107**, 201803 (2011);
    see also, D.G. Phillips II, "Search for the Rare Decay $K_L \to \pi^0\pi^0\mu^+\mu^-$," University of Virginia thesis, May 2009.
31. Y.C. Tung *et al.*, Phys. Rev. Lett. **102**, 051802 (2009).
32. M. Lu and M.B. Wise, Phys. Lett. **B324**, 461 (1994).
33. A.F. Falk, A. Lewandowski, and A.A. Petrov, Phys. Lett. **B505**, 107 (2001).
34. J. Charles *et al.* [CKMfitter Collab.], Phys. Rev. **D84**, 033005 (2011), updated results and plots available at: `http://ckmfitter.in2p3.fr`.
35. M. Bona *et al.* [UTfit Collab.] `arXiv:0707.0636`, `www.utfit.org/UTfit/`.
36. J. Brod, M. Gorbahn, and E. Stamou, Phys. Rev. **D83**, 034030 (2011).
37. F. Mescia and C. Smith, Phys. Rev. **D76**, 034017 (2007).
38. T. Inami and C.S. Lim, Prog. Theor. Phys. **65**, 297 (1981); Erratum Prog. Theor. Phys. **65**, 172 (1981).
39. G. Buchalla and A.J. Buras, Nucl. Phys. **B548**, 309 (1999);
    M. Misiak and J. Urban, Phys. Lett. **B451**, 161 (1999).
40. A.J. Buras *et al.*, Phys. Rev. Lett. **95**, 261805 (2005);
    A.J. Buras *et al.*, JHEP **0611**, 002 (2006).
41. J. Brod and M. Gorbahn, Phys. Rev. **D78**, 034006 (2008).
42. G. Isidori, F. Mescia, and C. Smith, Nucl. Phys. **B718**, 319 (2005).
43. S. Adler *et al.*, Phys. Rev. Lett. **88**, 041803 (2002).
44. S. Adler *et al.*, Phys. Rev. Lett. **84**, 3768 (2000).
45. S. Adler *et al.*, Phys. Lett. **B537**, 237 (2002).
46. A.V. Artamonov *et al.*, Phys. Rev. Lett. **101**, 191802 (2008).
47. G. Anelli *et al.*, CERN-SPSC-2005-013, 11 June 2005.
48. A.J. Buras, *et al.*, JHEP **1511**, 033 (2015).
49. L. Littenberg, Phys. Rev. **D39**, 3322 (1989).

50. G. Buchalla and G. Isidori, Phys. Lett. **B440**, 170 (1998).
51. Y. Grossman and Y. Nir, Phys. Lett. **B398**, 163 (1997).
52. J.K. Ahn *et al.*, Phys. Rev. **D81**, 072004 (2010).
53. J. Comfort *et al.*, "Proposal for $K_L^0 \to \pi^0 \nu\overline{\nu}$ Experiment at J-Parc," J-PARC Proposal 14 (2006), [http://koto.kek.jp/pub/p14.pdf].
54. J.K. Ahn *et al.*, Prog. Theor. Exp. Phys. 2017, 021C01.
55. K. Fuyuto, W. S. Hou, and M. Kohda, Phys. Rev. Lett. **114**, 171802 (2015).
56. G. D'Ambrosio and G. Isidori, Phys. Lett. **B530**, 108 (2002).
57. D. Bryman *et al.*, Int. J. Mod. Phys. **A21**, 487 (2006).
58. M. Blanke, PoS KAON**13**, 10(2013), [arXiv:1305.5671].
59. A. J. Buras *et al.*, JHEP **1511**, 166 (2015).
60. M. Bordone *et al.*, arXiv:1705.10729 [hep-ph].
61. L. Littenberg and G. Valencia, Phys. Lett. **B385**, 379 (1996);
C.-W. Chiang and F.J. Gilman, Phys. Rev. **D62**, 094026 (2000);
C.Q. Geng, I.J. Hsu, and Y.C. Lin, Phys. Rev. **D50**, 5744 (1994).
62. R. Ogata, *et al.*, Phys. Rev. **D84**, 052009 (2011).
63. S. Adler, *et al.*, Phys. Rev. **D63**, 032004 (2001).
64. D. Ambrose *et al.*, Phys. Rev. Lett. **84**, 1389 (2000).
65. G. Valencia, Nucl. Phys. **B517**, 339 (1998).
66. G. D'Ambrosio, G. Isidori, and J. Portoles, Phys. Lett. **B423**, 385 (1998).
67. G. Isidori and R. Unterdorfer, JHEP **0401**, 009 (2004).
68. D. Gomez-Dumm and A. Pich, Phys. Rev. Lett. **80**, 4633 (1998).
69. D. Ambrose *et al.*, Phys. Rev. Lett. **81**, 4309 (1998).
70. R. Aaij *et al.* [LHCb Collab.], [arXiv:1706.00758 [hep-ex]].
71. G. D'Ambrosio *et al.*, JHEP **9808**, 004 (1998);
C.O. Dib, I. Dunietz, and F.J. Gilman, Phys. Rev. **D39**, 2639 (1989).
72. G. Buchalla, G. D'Ambrosio, and G. Isidori, Nucl. Phys. **B672**, 387 (2003).
73. J.R. Batley *et al.*, Phys. Lett. **B576**, 43 (2003).
74. J.R. Batley *et al.*, Phys. Lett. **B599**, 197 (2004).
75. S. Friot, D. Greynat, and E. de Rafael, Phys. Lett. **B595**, 301 (2004).
76. G. Isidori, C. Smith, and R. Unterdorfer, Eur. Phys. J. **C36**, 57 (2004).
77. G. Ecker, A. Pich, and E. de Rafael, Phys. Lett. **237B**, 481 (1990);
L. Cappiello, G. D'Ambrosio, and M. Miragliuolo, Phys. Lett. **B298**, 423 (1993);
A. Cohen, G. Ecker, and A. Pich, Phys. Lett. **B304**, 347 (1993).
78. F. Gabbiani and G. Valencia, Phys. Rev. **D66**, 074006 (2002).
79. E. Abouzaid *et al.*, Phys. Rev. **D77**, 112004 (2008).
80. A. Lai *et al.*, Phys. Lett. **B536**, 229 (2002).
81. J. Donoghue and F. Gabbiani, Phys. Rev. **D56**, 1605 (1997).
82. E. Abouzaid *et al.*, Phys. Rev. **D76**, 052001 (2007).
83. W.M. Morse *et al.*, Phys. Rev. **D45**, 36 (1992).
84. A. Alavi-Harati *et al.*, Phys. Rev. **D64**, 012003 (2001).
85. H.B. Greenlee, Phys. Rev. **D42**, 3724 (1990).
86. A. Alavi-Harati *et al.*, Phys. Rev. Lett. **93**, 021805 (2004).
87. A. Alavi-Harati *et al.*, Phys. Rev. Lett. **84**, 5279 (2000).

88. M.V. Diwan, H. Ma, and T.L. Trueman, Phys. Rev. **D65**, 054020 (2002).
89. F. Mescia, C. Smith, and S. Trine, JHEP **0608**, 088 (2006).
90. R. Appel *et al.*, Phys. Rev. Lett. **83**, 4482 (1999);
    J.R. Batley *et al.*, Phys. Lett. **B677**, 246 (2009).
91. S.C. Adler *et al.*, Phys. Rev. Lett. **79**, 4756 (1997);
    R. Appel *et al.*, Phys. Rev. Lett. **84**, 2580 (2000);
    H.K. Park *et al.*, Phys. Rev. Lett. **88**, 111801 (2002);
    J.R. Batley *et al.*, Phys. Lett. **B697**, 107 (2011).
92. B. Bloch-Devaux *et al.* [NA48/2 Collab.], J. Phys. Conf. Ser. 800, 012029 (2017).
93. L. Cappiello *et al.*, Eur. Phys. J. **C72**, 1872 (2012) [Eur. Phys. J. **C72**, 2208 (2012)] [arXiv:1112.5184].
94. J. R. Batley *et al.* [NA48/1 Collab.], Phys. Lett. **B694**, 301 (2011).
95. C. Marin Benito *et al.* [LHCb Collab.] , J. Phys. Conf. Ser. 800, 012031 (2017).
96. G. Ecker, A. Pich, and E. de Rafael, Nucl. Phys. **B303**, 665 (1988).
97. G. D'Ambrosio and J. Portoles, Phys. Lett. **B386**, 403 (1996) [Phys. Lett. **B389**, 770 (1996)] [Erratum-ibid. B **395**, 390 (1997)] [hep-ph/9606213].
98. P. Kitching *et al.* [E787 Collab.], Phys. Rev. Lett. **79**, 4079 (1997) [hep-ex/9708011].
99. J.R. Batley *et al.*, Phys. Lett. **B730**, 141 (2014).
100. C. Lazzeroni *et al.*, Phys. Lett. **B732C**, 65 (2014).
101. A. Alavi-Harati *et al.*, Phys. Rev. Lett. **87**, 071801 (2001).
102. A. Abouzaid *et al.*, Phys. Rev. Lett. **99**, 051804 (2007).
103. V. Fanti *et al.*, Phys. Lett. **B458**, 458 (1999).
104. G. D'Ambrosio *et al.*, Eur. Phys. J. **C73**, 2678 (2013) [arXiv:1309.5736].
105. G. D'Ambrosio and T. Kitahara, [arXiv:1707.06999 [hep-ph]].