69. Review of Multibody Charm Analyses

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69.1 Overview

The study of multibody charm decays is a vibrant field, with vast and fast-increasing datasets, new developments in theory and experimental technique, and implications reaching beyond charm. This review is structured as follows: Sec. 69.2 summarizes key aspects of amplitude models that describe multibody charm decays, their application to data, and theory progress; Sec. 69.3 reviews the special role of charm threshold data in model-independent approaches; and Sec. 69.4 describes applications of multi-body charm analyses, focusing on Charge-Parity (*CP*) violation, charm mixing and the role of threshold data in this context. In Sec. 69.5, we conclude.

69.2 Kinematics & Models

The differential decay rate to a point $\mathbf{s} = (s_1, \ldots, s_n)$ in n dimensional phase space can be expressed as

$$d\Gamma = |\mathcal{M}(\mathbf{s})|^2 \left| \frac{\partial^n \phi}{\partial(s_1 \ldots s_n)} \right| d^n s \tag{69.1}$$

where $|\partial^n \phi / \partial(s_1 \ldots s_n)|$ represents the density of states at $\mathbf{s}$, and $\mathcal{M}$ the matrix element for the decay at that point in phase space, which is $2, 5, 8, \ldots$ dimensional for D decays to $3, 4, 5, \ldots$ spinless particles. Additional parameters are required to fully describe decays involving particles with non-zero spin in the initial or final state.

For the important case of D decays to three pseudoscalars, the decay kinematics can be represented in a two-dimensional Dalitz plot [1]. This is usually parameterized in terms of $s_{12} \equiv (p_1 + p_2)^2$ and $s_{23} \equiv (p_2 + p_3)^2$, where p_1, p_2, and p_3 are the four-momenta of the final-state particles. In terms of these variables, phase-space density is constant across the kinematically allowed region, so that any structure seen in the Dalitz plot is a direct consequence of the dynamics encoded in $|\mathcal{M}|^2$. Note that decays to four or more particles cannot be unambiguously described in terms of analogously-defined variables s_{ij} and s_{ijk}, which are all parity-even. Parity-odd kinematics are absent in three-body decays because the final-state momenta are confined to a plane; the use of parity-odd observables in four body decays is discussed in Sec. 69.4.4.

In the widely-used isobar approach, the matrix element $\mathcal{M}$ is modeled as a sum of interfering decay amplitudes, each proceeding through resonant two-body decays [2–6]. See [2, 7, 8] for a review of resonance phenomenology. In most analyses, each resonance is described by a relativistic Breit-Wigner [9] or Flatté [10] lineshape, and the model includes a non-resonant term with a constant phase and magnitude. This approach has well-known theoretical limitations, such as the violation of unitarity and analyticity, which can break the relationship between magnitude and phase across phase space. This motivates the use of more sophisticated descriptions, especially for broad, overlapping resonances (frequently found in S-wave components) where these limitations are particularly problematic. In charm analyses, these approaches have included the K-matrix [11] which respects two-body unitarity; the use of LASS scattering data [12]; dispersive methods [13–16]; multi-meson models using chiral Lagrangians [17–20]; and methods based on chiral unitarity [21–24]; QCD factorization [25–28]; and quasi model-independent parameterizations that use generic lineshapes, with minimal theory input and many free parameters, for a subset of resonances [29–37].

An important example for a multibody charm decay is $D^0 \to K^0_S \pi^+ \pi^-$, which is a key channel in *CP* violation and charm-mixing analyses. The latest analysis, published jointly by BaBar and Belle [38], describes the Dalitz plot with 1.1M $D^0 \to K^0_S \pi^+ \pi^-$ events with 18 resonant components, including four doubly Cabibbo suppressed ones. The model uses a K-matrix description for the

$\pi\pi$ S-wave based on [39] and input from LASS scattering data [12, 40] for the $K\pi$ S-wave, with no need to add a non-resonant component to describe the data. These data were re-analyzed by [25] in a QCD factorization framework, using line-shape parameterizations for the S [41, 42] and P wave [14] contributions that preserve two-body unitarity and analyticity. The analyses give compatible results for the components they share.

The field of charm amplitude analyses remains very active. Publications since the last update of this review two years ago include Dalitz plot analyses of $D^0 \to K^0_L\pi^+\pi^-$, $D^0 \to K^0_S\pi^+\pi^-$, $D^+_s \to K^0_S K^+\pi^0$, $D^+_s \to K^0_S K^0_S\pi^+$, $D^+_s \to K^+\pi^-\pi^+$, $D^+_s \to \pi^+\pi^0\eta'$, and $D^+_s \to \pi^+\pi^0\pi^0$ by BESIII [43–48]; $D^+_s \to \pi^+\pi^-\pi^+$ by LHCb [36] and BESIII [35]; and $D^+ \to \pi^+\pi^-\pi^+$ by LHCb [37]. LHCb studied the amplitude structure of $\Lambda_c \to pK^-\pi^+$ [49] and BESIII that of $\Lambda_c \to \Lambda\pi^+\pi^0$ [50]. BESIII continued its series of four-body amplitude analyses with $D^+ \to K^0_S\pi^+\pi^0\pi^0$ and $D^+_s \to K^+\pi^-\pi^+\pi^0$ [51, 52], and performed a five-body amplitude analysis of $D^+_s \to K^+K^-\pi^+\pi^+\pi^-$ [53]. These analyses provide valuable insight into hadronic dynamics and have wider impact beyond that. The amplitude structure of $\Lambda_c \to pK^-\pi^+$ for example is found to have a high sensitivity to the Λ_c polarization, which can prove valuable in polarization measurements of Λ_b baryons decaying to Λ_c [49, 54]. The importance of charm amplitude analyses for CP violation and charm mixing is further discussed below. Comparing the $D^0 \to K^0_L\pi^+\pi^-$ amplitude structure with that of $D^0 \to K^0_S\pi^+\pi^-$ reveals U-spin breaking effects [43]. The D^+_s and $D^+ \to \pi^+\pi^-\pi^+$ analyses mentioned above, as well an earlier one by BaBar [31], analyse the $\pi^+\pi^-$ S-wave using a quasi-model-independent approach. They observe substantial differences between the $\pi^+\pi^+$ S-wave components observed in the two decays, and between the associated phase shift seen in either compared to that observed in scattering data [55–57]. These observations suggest that broad scalar meson resonances in these decays might be generated from meson-meson rescattering [37]; more broadly, they emphasize the importance of final state interactions in multibody decays.

Noteworthy is the increasing sophistication of recent amplitude analyses, most of which go substantially beyond the isobar model with Breit Wigner and Flatté lineshapes. However, with the notable exceptions of [15, 16, 23, 24, 58, 59], they remain within the isobar framework which describes the decay as a series of two-body processes, ignores long-range hadronic effects such as re-scattering, and does not respect three (or four)-body unitarity and analyticity.

Several groups work on improved models. Dispersive techniques, which respect three-body unitarity and analyticity, have been applied to regions of the $D^+ \to K^-\pi^+\pi^+$ and $D^+ \to K^0_S\pi^0\pi^+$ Dalitz plots below the $\eta' K$ threshold [15, 16], where they provide a good description of the data with fewer fit parameters than the isobar approach. Reference [58] uses a unitary coupled channel approach to describe $D^+ \to K^-\pi^+\pi^+$, which has no restrictions on the kinematic range, but requires additional parameters to describe the Dalitz plot above the $\eta' K$ threshold. Using an effective chiral Lagrangian, the authors of reference [19] provide a description of the annihilation contribution to the decay amplitude which respects three-body unitarity. The approach provides a good description of LHCb's $D^+ \to K^+K^-K^+$ data, with fewer parameters than an equivalent isobar model [59]. The same channel has more recently been re-analyzed by the authors of [23], who argue that the internal emission diagram should dominate and use a chiral unitarity-based approach to achieve a reasonable description of the data with two free parameters.

Limitations in the theoretical description of interfering resonances are the leading source of systematic uncertainty in many analyses. In some cases, the model uncertainty can be removed through model-independent methods, often relying on input from the charm threshold, as discussed in Sec. 69.3. The authors of [60] expand the scope and applicability of the quasi model-independent approach in amplitude fits. At the same time, increasingly sophisticated models are being developed, and applied to data. The authors of [20] and [61] provide valuable practical frameworks for sophisticated amplitude analyses.

69.3 Model Independent Methods and the Charm Threshold

Precision measurements of the CP-violation phase γ/ϕ_3 using $B^- \to DK^-$, $D \to K^0_S\pi^+\pi^-$ and related decay modes, as well as precision measurements of charm-mixing parameters in decay modes such as $D^0 \to K^0_S\pi^+\pi^-$, require input on the relative decay amplitudes of D^0 and $\overline{D}^0$. In the above examples, $K^0_S\pi^+\pi^-$ can be replaced with other final state accessible to both D^0 and $\overline{D}^0$ [62–70]. While the magnitudes of the decay amplitudes can be measured precisely using the vast charm samples at the B factories and LHCb, obtaining the relative phases requires either amplitude models with reliable phase motion, or model-independent approaches. More details on the measurements of γ/ϕ_3 and charm mixing can be found in sections 69.4.1 and 69.4.2.

Model-independent measurements of relative phases between D^0 and $\overline{D}^0$ decay amplitudes rely on interference effects in the decays of well-defined coherent superpositions of D^0 and $\overline{D}^0$. These are accessible at the charm threshold [65, 68, 70–75], where CLEO–c and BESIII operate. There, quantum-correlated D-meson pairs are produced, which have opposite CP and flavour content. For example if one (the tag) is identified as a CP eigenstate through its decay (e.g. to K^+K^- or $K_S\pi^0$) the other D meson is an eigenstate state with the opposite CP eigenvalue. The decay rate of a CP eigenstate $\frac{1}{\sqrt{2}}\left(D^0 \pm \overline{D}^0\right)$ is proportional to $|A_D|^2 + |\overline{A}_D|^2 \pm 2|A_D||\overline{A}_D|\cos(\delta_D)$, where A_D and $\overline{A}_D$ are the D^0 and $\overline{D}^0$ decay amplitudes, and δ_D is their sought-after phase difference. Other quantum correlations give access to $\sin(\delta_D)$. For multibody decays, A_D, $\overline{A}_D$ and δ_D vary across phase space. Model-independent phase information is measured either integrated over the entire phase space of the decay, or in sub-regions/bins. The results can be expressed in terms of one complex parameter $\mathcal{Z} = Re^{-i\delta} = c + is$ per pair of CP-conjugate bins; c and s are the average of $\cos(\delta_D)$ and $\sin(\delta_D)$ over that region of phase space. Amplitude models can be used to optimize the binning for sensitivity to γ/ϕ_3, without introducing a model-dependent bias in the result [76]; novel unbinned approaches have the potential to further increase the precision on γ/ϕ_3 in future analyses [77–79].

CLEO–c data have been analyzed to provide binned $\mathcal{Z}$ for the self-conjugate decays $D^0 \to K^0_{S,L}\pi^+\pi^-$, $K^0_{S,L}K^+K^-$, $\pi^+\pi^-\pi^+\pi^-$, and $K^0_S\pi^-\pi^+\pi^0$ and phase space-integrated values for $D^0, \overline{D}^0 \to K^0_S K^+\pi^-$, $K^+\pi^-\pi^0$ and $K^+\pi^-\pi^+\pi^+$ [80–85]. BESIII significantly improved the precision for the $K^0_S\pi^+\pi^-$, $K^0_S K^+K^-$, $K^+\pi^-\pi^0$ and $K^+\pi^-\pi^+\pi^-$ final states [86–89]. CLEO–c and BESIII $K^+\pi^-\pi^+\pi^-$ data were also analysed in four bins of five-dimensional phase space [89, 90] with a binning based on [33].

For self-conjugate decays such as $D^0 \to \pi^+\pi^-\pi^0$, one can define the CP-even fraction F_+ which is $+1$ for a CP-even eigenstate and 0 for a CP-odd one [73]. F_+ is related to $Re(\mathcal{Z}) = c$, defined for a single CP-conjugate bin pair. For the purpose of γ/ϕ_3 measurements, this information allows the decay to be treated equivalently to a two-body decay, where γ/ϕ_3 is determined from relative decay rates in multiple decay modes, without the need to resolve the phase-space distribution. In the mathematical expressions for the decay rates, a factor $(2F_+ - 1)$ multiplies the γ/ϕ_3-sensitive term, leading to best sensitivity for $F_+ = 1$ or $F_+ = 0$. $D^0 \to \pi^+\pi^-\pi^0$ is found to be, with $F_+ = (97.3 \pm 1.7)\%$, compatible with being completely CP-even (CLEO–c data); a recent BESIII analysis of $D^0 \to K_{L,S}\pi^+\pi^-$ finds $F_+ = (35.3 \pm 0.6 \pm 1.4)\%$ and $(55.6 \pm 0.6 \pm 1.2)\%$, respectively; $D^0 \to \pi^+\pi^-\pi^+\pi^-$ (BESIII, CLEO–c data), $D^0 \to K^+K^-\pi^+\pi^-$ (BESIII), $D^0 \to K^+K^-\pi^0$ (CLEO–c data), are all predominantly CP even with F_+ ranging from 73% to 77%; $D^0 \to K^0_S\pi^+\pi^-\pi^0$ (BESIII, CLEO–c data), is predominantly CP odd with $F_+ = (23.5 \pm 10)\%$ [43, 74, 82, 83, 91–93]. In cases where results from both BESIII and CLEO–c data exist, BESIII's more precise results are quoted.

Comparing measured parameters like F_+ and $\mathcal{Z}$ with those predicted from amplitude models provides a test of the models' phase motion. Model-predicted values for F_+ in $D^0 \to K_{L,S}\pi^+\pi^-$ agree, with $(35.3 \pm 0.6 \pm 1.4)\%$ and $(55.6 \pm 0.6 \pm 1.2)\%$ [38, 43] well with the measured values given above; as does the measured F_+ for $D^0 \to \pi^+\pi^-\pi^+\pi^-$, $F_+ = 0.769 \pm 0.023$ [92], with 0.729 ± 0.020 calculated from [32]; and the coherence factor $|\mathcal{Z}| = R$ measured in $D^0 \to K^\pm\pi^\mp\pi^\pm\pi^\mp$, $0.44^{+0.10}_{-0.09}$ [85, 94], with 0.459 ± 0.025 predicted by [33]. The latest bin-wise comparisons between model and data for $D^0 \to K_{L,S}\pi^+\pi^-$, $K^\pm\pi^\mp\pi^\pm\pi^\mp$, $K^\pm\pi^\mp\pi^0$, $K^0_S K^+K^-$, and $\pi^+\pi^-\pi^+\pi^-$ can be found in [43, 82, 86–89, 89] and generally show good agreement. These results are a welcome surprise given the preceding discussion on the theoretical shortcomings of amplitude models.

69.4 Applications of multibody charm analyses for *CP* violation and mixing

Amplitude analyses provide sensitivity to both relative magnitudes and phases of the interfering decay amplitudes. It is especially the sensitivity to phases that makes amplitude analyses such a uniquely powerful tool for studying a wide range of phenomena. Here we concentrate on their critical role in *CP* violation and mixing measurements. Properties of light-meson resonances determined in *D* amplitude analyses are reported in the light-unflavored-meson section of this *Review*.

The closely related topics of multibody charm decays in measurements of *CP* violation in beauty decays to charm and charm mixing will be discussed in turn, followed by a review of searches for time-integrated *CP* violation in multibody charm decays.

69.4.1 *CP violation in decays of Beauty to Charm*

Neutral D mesons originating from $B^- \to DK^-$ (here denoted as D_{B^-}) are a superposition of D^0 and $\overline{D}^0$ with a relative phase that depends on the CKM unitarity triangle parameter γ/ϕ_3,

$$D_{B^-} \propto D^0 + r_B e^{i(\delta_B - \gamma)} \overline{D}^0,$$

where δ_B is a *CP* conserving strong phase, and $r_B \sim 0.1$ is the magnitude of the ratio of the $B^- \to \overline{D}^0 K^-$ and $B^- \to D^0 K^-$ decay amplitudes. In the corresponding *CP*-conjugate expression, γ/ϕ_3 changes sign. The amplitude analysis of the subsequent $D_{B^\pm}$ decay to a state accessible to both D^0 and $\overline{D}^0$ allows the measurement of γ/ϕ_3 [62–69]. The method generalizes to similar B hadron decays, such as $B^0 \to DK^{*0}$, and, albeit with reduced sensitivity, to decays where the kaon is replaced by a pion.

Measurements based on this technique have been reported by BaBar [95, 96], Belle [97–99], Belle II [100], and LHCb [101–114]. Since it relies on $D^0 - \overline{D}^0$ interference, the phase differences between the relevant D^0 and $\overline{D}^0$ decay amplitudes are required in order to extract γ/ϕ_3. Because the theoretical shortcomings of amplitude models discussed above make their phases unreliable, most recent γ/ϕ_3 measurements use amplitude model-independent approaches to obtain this information [98–105, 107–114]. These depend critically on information from the charm threshold described in Sec. 69.3. Charm mixing also results in a (time-dependent) $D^0 - \overline{D}^0$ superposition, that can be used to measure the relevant phase information as input to γ/ϕ_3 measurements. This method is particularly powerful in doubly Cabibbo-suppressed decays such as $D^0 \to K^+\pi^-\pi^+\pi^-$, and when used in combination with threshold data [115, 116]. Under some circumstances, with large data sets, the relevant strong phases and γ/ϕ_3 can be extracted simultaneously without external input, for example in simultaneous analysis of the $B^0 \to DK^+\pi^-$ Dalitz plot and that of the subsequent $D \to K^0_{\mathrm{S}}\pi^+\pi^-$ decay [117]. However, the global effort to measure γ/ϕ_3 to sub-degree precision will continue to rely critically on input from the charm threshold and BESIII's increasing datasets [118].

The most precise individual γ/ϕ_3 measurement with an uncertainty of $\sim 5^\circ$ results from LHCb's amplitude model-independent study of $D_{B^-} \to K^0_S\pi^+\pi^-$ and $D_{B^-} \to K^0_S K^+K^-$ [119], using input from the charm threshold [80, 87]. LHCb's analysis of $D_{B^-} \to K^+\pi^-\pi^+\pi^-$ decays leads to the second most precise individual γ/ϕ_3 measurement with $\gamma/\phi_3 = 55.8^{\circ}{}^{+6.0^\circ}_{-5.8^\circ} \pm 0.6^{\circ}{}^{+6.7^\circ}_{-4.3^\circ}$ [113]. Here, the first uncertainty is due to $B^\pm \to DK^\pm$ statistics, the second systematic, and the third due to the uncertainty on charm-threshold input. This illustrates the importance of charm-threshold data, and the need for the analysis of larger threshold datasets to keep pace with the sharp increase in $B^\pm \to DK^\pm$ data expected at the upgraded LHCb detector and Belle II. Further improvements in this mode are expected from including information on relative phases between D^0 and $\overline{D}^0$ amplitudes obtained from charm mixing [85, 94, 116].

The current world average on γ/ϕ_3 is dominated by LHCb who finds, combining its results with input from CLEO–c and BESIII in an amplitude-model independent approach, $\gamma = 63.8^\circ{}^{+3.5^\circ}_{-3.7^\circ}$ [111, 120]. Belle II, which is expected to reach excellent precision in γ/ϕ_3, recently released its first γ/ϕ_3 result, based on Belle, Belle II and charm threshold data [100].

The interference between mixing and decay in $B^0 \to D^0 h^0$ with $h^0 = \pi^0, \eta, \omega$ provides sensitivity to β, which can be extracted from the Dalitz plot of the subsequent $D^0 \to K^0_S\pi^+\pi^-$ decay [38, 121–124]. The combined BaBar/Belle analysis based on this technique resolved the ambiguity in β present in other measurements [38].

Further details on *CP* violation in beauty and charm can be found in [125, 126].

69.4.2 *Charm Mixing and CP violation*

Time-dependent amplitude analyses in decays to final states that are accessible to both D^0 and $\overline{D}^0$ have unique sensitivity to mixing parameters. A Dalitz plot analysis of a self-conjugate final state, such as $K^0_S\pi^+\pi^-$ and $K^0_S K^+K^-$, allows a direct measurement of x and y, the normalized mass and width difference of the $D^0 - \overline{D}^0$ system's mass eigenstates [127]. This is in contrast to decays like $D^0 \to K^+\pi^-$which only provide access to decay-specific parameters x'^2, y'.

The phase differences between D^0 and $\overline{D}^0$ amplitudes across the Dalitz plot affect these measurements in the same way as those of γ/ϕ_3 in $B^+ \to DK^+$, and can be taken into account in an amplitude model-independent way using the same charm threshold results [65, 70–75, 125]. This approach recently resulted in the first observation of a non-zero mass difference between the neutral charm mass eigenstates by LHCb [128], using a model-independent analysis of $D^0 \to K^0_S\pi^+\pi^-$. The "bin flip" technique applied in this measurements exploits that *CP*-conjugate regions of the $D^0 \to K^0_S\pi^+\pi^-$ Dalitz plot have near-identical experimental efficiencies [129]. This measurement is also sensitive to *CP* violation in mixing and in the interference between mixing and decay, which is discussed further in [125, 126].

69.4.3 *Everything, Everywhere, All at Once*

The discussion above shows how closely related the measurements of γ/ϕ_3, charm mixing, and the analysis of charm threshold data are, due to the shared charm parameters. While above, the emphasis was on the importance of charm as input to γ/ϕ_3, and threshold data as input to charm mixing and γ/ϕ_3, these relationships are in fact omnidirectional. This is demonstrated by LHCb's combined γ/ϕ_3 fit to γ/ϕ_3-sensitive beauty decays to charm such as $B^- \to DK^-$, charm mixing, and charm threshold data. This combined analysis has a notable effect on charm parameters due to the input from beauty decays to charm [111].

69.4.4 *Searches for time-integrated CPV in charm*

The recent observation of direct *CP* violation in LHCb's analysis of $D^0 \to \pi^+\pi^-$ and $D^0 \to K^+K^-$ decays [130, 131], and the open question as to its Standard Model or beyond the Standard Model nature, add renewed interest in the search for *CP* violation in other decays.

Multibody decays, with their rich structure and varying strong phase across phase space, could potentially have particularly high sensitivity to *CP* violation. Comparing the results of amplitude fits for *CP*-conjugate decay modes provides a measure of *CP* violation, as for example done for $D^0 \to K^0_S K^\pm\pi^\mp$, $D^0 \to K^+K^-\pi^+\pi^-$, and $D^0 \to \pi^+\pi^-\pi^+\pi^-$ using LHCb and CLEO–c data [32, 132–134]. A widely-used amplitude model-independent technique to search for local *CP* violation in multibody phase space is based on performing a χ^2 comparison of *CP*-conjugate phase-space distributions. This method was pioneered by BaBar [135] and developed further in [136–138] with results reported by BaBar [135,139] and LHCb in $D^\pm \to K^+K^-\pi^\pm$ [140,141], CDF in $D^0 \to K^0_S\pi^+\pi^-$ [142], and LHCb in $D^+_{(s)} \to K^+K^-K^+$, $D^+ \to \pi^-\pi^+\pi^+$, $D^0 \to K^+K^-\pi^+\pi^-$, $D^0 \to \pi^+\pi^-\pi^+\pi^-$, and $\Xi^+_c \to pK^-\pi^+$ [138, 143–145]. Unbinned methods can increase sensitivity [146,147] and have been applied by LHCb to $D^+ \to \pi^-\pi^+\pi^+$, $D^0 \to \pi^+\pi^-\pi^0$, $D^0 \to \pi^+\pi^-\pi^+\pi$, and $\Xi^+_c \to pK^-\pi^+$ [144, 145, 148–150].

An alternative model-independent approach is based on observables in four body decays that are odd under motion reversal ("naïve T") [151–159], which is equivalent to P for scalar particles [159]. One such observable is $C_T = \vec{p}_2 \cdot (\vec{p}_3 \times \vec{p}_4)$ where $\vec{p}_i$ are the decay products' three momenta in the decay's center of mass frame. Comparing the P violating asymmetry $A_T \equiv \frac{\Gamma(C_T>0)-\Gamma(C_T<0)}{\Gamma(C_T>0)+\Gamma(C_T<0)}$ with its C-conjugate provides sensitivity to CP violation. Additional variables are proposed in [159], which also provides an excellent review of this topic. Searches for CP violation using P-odd variables have been carried out by BaBar, Belle, FOCUS, and LHCb in $D^0 \to K^+K^-\pi^+\pi^-$ [160–163], $D^+ \to K^+K^0_S\pi^+\pi^-$, $D^+_s \to K^+K^0_S\pi^+\pi^-$ [164], $D^0 \to K^0_S\pi^+\pi^-\pi^0$ [165], and $D^0 \to \pi^+\pi^-\pi^+\pi^-$ [149].

Recently, LHCb performed the first full angular analysis of the rare decays $D^0 \to \pi^+\pi^-\mu^+\mu^-$ and $D^0 \to K^+K^-\mu^+\mu^-$. From this, several observables with sensitivity to physics beyond the Standard Model are constructed, including CP-violating ones. The results are in agreement with Standard-Model predictions [166, 167].

All results of analyses mentioned in this section are compatible with CP conservation. The observation of CP violation in two body charm decays [130, 131], and the vast data samples about to be collected, provide grounds for optimism that this may change in the foreseeable future.

69.5 Summary

Multibody charm decays offer a rich phenomenology, including unique sensitivity to CP violation and charm mixing. This is a highly dynamic field with many new results (some of which we presented here) and rapidly increasing, high quality datasets. These datasets constitute a huge opportunity, but also a challenge to improve the theoretical descriptions of soft hadronic effects in multibody decays. For some measurements, model-independent methods, many relying on input from the charm threshold, provide a way of removing model-induced uncertainties. At the same time, substantial progress in the theoretical description of multibody decays is being made.

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