

# 1. THE CKM QUARK-MIXING MATRIX

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## 1.1. Introduction

The masses and mixings of quarks have a common origin in the Standard Model (SM). They arise from the Yukawa interactions with the Higgs condensate,

$$\mathcal{L}_Y = -Y_{ij}^d \overline{Q_{Li}^I} \phi d_{Rj}^I - Y_{ij}^u \overline{Q_{Li}^I} \epsilon \phi^* u_{Rj}^I + \text{h.c.}, \quad (1.1)$$

where  $Y^{u,d}$  are  $3 \times 3$  complex matrices,  $\phi$  is the Higgs field,  $i, j$  are generation labels, and  $\epsilon$  is the  $2 \times 2$  antisymmetric tensor.  $Q_L^I$  are left-handed quark doublets, and  $d_R^I$  and  $u_R^I$  are right-handed down- and up-type quark singlets, respectively, in the weak-eigenstate basis. When  $\phi$  acquires a vacuum expectation value,  $\langle \phi \rangle = (0, v/\sqrt{2})$ , Eq. (1.1) yields mass terms for the quarks. The physical states are obtained by diagonalizing  $Y^{u,d}$  by four unitary matrices,  $V_{L,R}^{u,d}$ , as  $M_{\text{diag}}^f = V_L^f Y^f V_R^{f\dagger} (v/\sqrt{2})$ ,  $f = u, d$ . As a result, the charged-current  $W^\pm$  interactions couple to the physical  $u_{Lj}$  and  $d_{Lk}$  quarks with couplings given by

$$\frac{-g}{\sqrt{2}} (\overline{u_L}, \overline{c_L}, \overline{t_L}) \gamma^\mu W_\mu^+ V_{\text{CKM}} \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} + \text{h.c.}, \quad V_{\text{CKM}} \equiv V_L^u V_L^{d\dagger} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \quad (1.2)$$

This Cabibbo-Kobayashi-Maskawa (CKM) matrix [1,2] is a  $3 \times 3$  unitary matrix. It can be parameterized by three mixing angles and the  $CP$ -violating KM phase [2]. Of the many possible conventions, a standard choice has become [3]

$$V_{\text{CKM}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \quad (1.3)$$

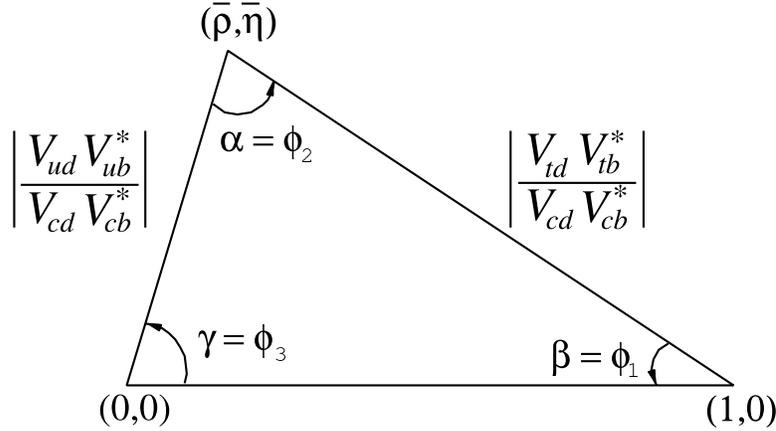
where  $s_{ij} = \sin \theta_{ij}$ ,  $c_{ij} = \cos \theta_{ij}$ , and  $\delta$  is the phase responsible for all  $CP$ -violating phenomena in flavor-changing processes in the SM. The angles  $\theta_{ij}$  can be chosen to lie in the first quadrant, so  $s_{ij}, c_{ij} \geq 0$ .

It is known experimentally that  $s_{13} \ll s_{23} \ll s_{12} \ll 1$ , and it is convenient to exhibit this hierarchy using the Wolfenstein parameterization. We define [4–6]

$$s_{12} = \lambda = \frac{|V_{us}|}{\sqrt{|V_{ud}|^2 + |V_{us}|^2}}, \quad s_{23} = A\lambda^2 = \lambda \left| \frac{V_{cb}}{V_{us}} \right|, \\ s_{13}e^{i\delta} = V_{ub}^* = A\lambda^3 (\rho + i\eta) = \frac{A\lambda^3 (\bar{\rho} + i\bar{\eta}) \sqrt{1 - A^2\lambda^4}}{\sqrt{1 - \lambda^2} [1 - A^2\lambda^4 (\bar{\rho} + i\bar{\eta})]}. \quad (1.4)$$

These relations ensure that  $\bar{\rho} + i\bar{\eta} = -(V_{ud}V_{ub}^*)/(V_{cd}V_{cb}^*)$  is phase-convention-independent, and the CKM matrix written in terms of

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**Figure 1.1:** Sketch of the unitarity triangle.

$\lambda$ ,  $A$ ,  $\bar{\rho}$ , and  $\bar{\eta}$  is unitary to all orders in  $\lambda$ . The definitions of  $\bar{\rho}$ ,  $\bar{\eta}$  reproduce all approximate results in the literature. For example,  $\bar{\rho} = \rho(1 - \lambda^2/2 + \dots)$  and we can write  $V_{\text{CKM}}$  to  $\mathcal{O}(\lambda^4)$  either in terms of  $\bar{\rho}$ ,  $\bar{\eta}$  or, traditionally,

$$V_{\text{CKM}} = \begin{pmatrix} 1 - \lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \lambda^2/2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4). \quad (1.5)$$

The CKM matrix elements are fundamental parameters of the SM, so their precise determination is important. The unitarity of the CKM matrix imposes  $\sum_i V_{ij} V_{ik}^* = \delta_{jk}$  and  $\sum_j V_{ij} V_{kj}^* = \delta_{ik}$ . The six vanishing combinations can be represented as triangles in a complex plane, of which the ones obtained by taking scalar products of neighboring rows or columns are nearly degenerate. The areas of all triangles are the same, half of the Jarlskog invariant,  $J$  [7], which is a phase-convention-independent measure of  $CP$  violation, defined by  $\text{Im}[V_{ij} V_{kl} V_{il}^* V_{kj}^*] = J \sum_{m,n} \varepsilon_{ikm} \varepsilon_{jln}$ .

The most commonly used unitarity triangle arises from

$$V_{ud} V_{ub}^* + V_{cd} V_{cb}^* + V_{td} V_{tb}^* = 0, \quad (1.6)$$

by dividing each side by the best-known one,  $V_{cd} V_{cb}^*$  (see Fig. 1). Its vertices are exactly  $(0, 0)$ ,  $(1, 0)$ , and, due to the definition in Eq. (1.4),  $(\bar{\rho}, \bar{\eta})$ . An important goal of flavor physics is to overconstrain the CKM elements, and many measurements can be conveniently displayed and compared in the  $\bar{\rho}$ ,  $\bar{\eta}$  plane.

Processes dominated by loop contributions in the SM are sensitive to new physics, and can be used to extract CKM elements only if the SM is assumed. We describe such measurements assuming the SM in Sec. 1.2 and 1.3, give the global fit results for the CKM elements in Sec. 1.4, and discuss implications for new physics in Sec. 1.5.

## 1.2. Magnitudes of CKM elements

### 1.2.1. $|V_{ud}|$ :

The most precise determination of  $|V_{ud}|$  comes from the study of superallowed  $0^+ \rightarrow 0^+$  nuclear beta decays, which are pure vector transitions. Taking the average of the twenty most precise determinations [8] yields

$$|V_{ud}| = 0.97425 \pm 0.00022. \quad (1.7)$$

The error is dominated by theoretical uncertainties stemming from nuclear Coulomb distortions and radiative corrections. A precise determination of  $|V_{ud}|$  is also obtained from the measurement of the neutron lifetime. The theoretical uncertainties are very small, but the determination is limited by the knowledge of the ratio of the axial-vector and vector couplings,  $g_A = G_A/G_V$  [9]. The PIBETA experiment [10] has improved the measurement of the  $\pi^+ \rightarrow \pi^0 e^+ \nu$  branching ratio to 0.6%, and quote  $|V_{ud}| = 0.9728 \pm 0.0030$ , in agreement with the more precise result listed above. The interest in this measurement is that the determination of  $|V_{ud}|$  is very clean theoretically, because it is a pure vector transition and is free from nuclear-structure uncertainties.

### 1.2.2. $|V_{us}|$ :

The product of  $|V_{us}|$  and the form factor at  $q^2 = 0$ ,  $|V_{us}| f_+(0)$ , has been extracted traditionally from  $K_L^0 \rightarrow \pi e \nu$  decays in order to avoid isospin-breaking corrections ( $\pi^0 - \eta$  mixing) that affect  $K^\pm$  semileptonic decay, and the complications induced by a second (scalar) form factor present in the muonic decays. The last round of measurements has led to enough experimental constraints to justify the comparison between different decay modes. Systematic errors related to the experimental quantities, *e.g.*, the lifetime of neutral or charged kaons, and the form factor determinations for electron and muonic decays, differ among decay modes, and the consistency between different determinations enhances the confidence in the final result. For this reason, we follow the prescription [11] to average  $K_L^0 \rightarrow \pi e \nu$ ,  $K_L^0 \rightarrow \pi \mu \nu$ ,  $K^\pm \rightarrow \pi^0 e^\pm \nu$ ,  $K^\pm \rightarrow \pi^0 \mu^\pm \nu$  and  $K_S^0 \rightarrow \pi e \nu$ . The average of these five decay modes yields  $|V_{us}| f_+(0) = 0.2163 \pm 0.0005$ . Results obtained from each decay mode, and exhaustive references to the experimental data, are listed for instance in Ref. [9]. The form factor value  $f_+(0) = 0.960^{+0.005}_{-0.006}$  [12] from a three-flavor unquenched lattice QCD calculation gives [9]  $|V_{us}| = 0.2253 \pm 0.0014$ .<sup>1</sup> The broadly used classic calculation of  $f_+(0)$  [15] is in good agreement with this value, while other calculations [16] differ by as much as 2%.

The calculation of the ratio of the kaon and pion decay constants enables one to extract  $|V_{us}/V_{ud}|$  from  $K \rightarrow \mu \nu(\gamma)$  and  $\pi \rightarrow \mu \nu(\gamma)$ , where  $(\gamma)$  indicates that radiative decays are included [17]. The KLOE measurement of the  $K^+ \rightarrow \mu^+ \nu(\gamma)$  branching ratio [18], combined

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<sup>1</sup> For lattice QCD inputs, we use the averages from Ref. 13 whenever possible, unless the minireviews [9,14] choose other values. Hereafter, the first error is statistical and the second is systematic, unless mentioned otherwise.

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with the lattice QCD calculation,  $f_K/f_\pi = 1.1947 \pm 0.0045$  [19], leads to  $|V_{us}| = 0.2253 \pm 0.0010$ , where the accuracy is limited by the knowledge of the ratio of the decay constants. The average of these two determinations is quoted by Ref. 9 as

$$|V_{us}| = 0.2253 \pm 0.0008. \quad (1.8)$$

The latest determination from hyperon decays can be found in Ref. 21. The authors focus on the analysis of the vector form factor, protected from first order  $SU(3)$  breaking effects by the Ademollo-Gatto theorem [22], and treat the ratio between the axial and vector form factors  $g_1/f_1$  as experimental input, thus avoiding first order  $SU(3)$  breaking effects in the axial-vector contribution. They find  $|V_{us}| = 0.2250 \pm 0.0027$ , although this does not include an estimate of the theoretical uncertainty due to second-order  $SU(3)$  breaking, contrary to Eq. (1.8). Concerning hadronic  $\tau$  decays to strange particles, the latest determinations based on LEP, BABAR, and Belle data yield  $|V_{us}| = 0.2202 \pm 0.0015$  [23]. A measurement of the ratio of branching fractions  $\mathcal{B}(\tau \rightarrow K\nu)/\mathcal{B}(\tau \rightarrow \pi\nu)$  by BABAR [24] combined with the above  $f_K/f_\pi$  value gives  $|V_{us}| = 0.2244 \pm 0.0024$ .

### 1.2.3. $|V_{cd}|$ :

The magnitude of  $V_{cd}$  can be extracted from semileptonic charm decays, using theoretical knowledge of the form factors. In semileptonic  $D$  decays, unquenched lattice QCD calculations have predicted the normalization of the  $D \rightarrow \pi\ell\nu$  and  $D \rightarrow K\ell\nu$  form factors [13]. The dependence on the invariant mass of the lepton pair,  $q^2$ , is determined from lattice QCD and theoretical constraints from analyticity [14]. Using three-flavor unquenched lattice QCD calculations for  $D \rightarrow \pi\ell\nu$ ,  $f_+^{D\pi}(0) = 0.666 \pm 0.029$  [13], and the average of recent CLEO-c [25] and Belle [26] measurements of  $D \rightarrow \pi\ell\nu$  decays, one obtains  $|V_{cd}| = 0.220 \pm 0.006 \pm 0.010$ , where the first uncertainty is experimental, and the second is from the theoretical uncertainty of the form factor.

Earlier determinations of  $|V_{cd}|$  came from neutrino scattering data. The difference of the ratio of double-muon to single-muon production by neutrino and antineutrino beams is proportional to the charm cross section off valence  $d$  quarks, and therefore to  $|V_{cd}|^2$  times the average semileptonic branching ratio of charm mesons,  $\mathcal{B}_\mu$ . The method was used first by CDHS [27] and then by CCFR [28,29] and CHARM II [30]. Averaging these results is complicated, because it requires assumptions about the scale of the QCD corrections, and because  $\mathcal{B}_\mu$  is an effective quantity, which depends on the specific neutrino beam characteristics. Given that no recent experimental input is available, we quote the average from a past review,  $\mathcal{B}_\mu|V_{cd}|^2 = (0.463 \pm 0.034) \times 10^{-2}$  [31]. Analysis cuts make these experiments insensitive to neutrino energies smaller than 30 GeV. Thus,  $\mathcal{B}_\mu$  should be computed using only neutrino interactions with visible energy larger than 30 GeV. An appraisal [32] based on charm-production fractions measured in neutrino interactions [33,34] gives  $\mathcal{B}_\mu = 0.088 \pm 0.006$ . Data from the CHORUS experiment [35] are sufficiently precise to extract  $\mathcal{B}_\mu$  directly, by comparing the number of charm decays with a muon to the total number of charmed hadrons

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found in the nuclear emulsions. Requiring the visible energy to be larger than 30 GeV, CHORUS finds  $\mathcal{B}_\mu = 0.085 \pm 0.009 \pm 0.006$ . We use the average of these two determinations,  $\mathcal{B}_\mu = 0.087 \pm 0.005$ , and obtain  $|V_{cd}| = 0.230 \pm 0.011$ . Averaging the two determinations above, we find

$$|V_{cd}| = 0.225 \pm 0.008. \quad (1.9)$$

### 1.2.4. $|V_{cs}|$ :

The direct determination of  $|V_{cs}|$  is possible from semileptonic  $D$  or leptonic  $D_s$  decays, using unquenched lattice QCD calculations of the semileptonic  $D$  form factor or the  $D_s$  decay constant. For muonic decays, the average of Belle [36], CLEO-c [37] and BABAR [38] is  $\mathcal{B}(D_s^+ \rightarrow \mu^+ \nu) = (5.56 \pm 0.24) \times 10^{-3}$  [39]. For decays to  $\tau$  leptons, the average of CLEO-c [37,40,41], BABAR [38] and Belle [36] gives  $\mathcal{B}(D_s^+ \rightarrow \tau^+ \nu) = (5.56 \pm 0.22) \times 10^{-2}$  [39]. From each of these values, determinations of  $|V_{cs}|$  can be obtained using the PDG values for the mass and lifetime of the  $D_s$ , the masses of the leptons, and  $f_{D_s} = (248.6 \pm 2.7) \text{ MeV}$  [13]. The average of these determinations gives  $|V_{cs}| = 1.008 \pm 0.021$ , where the error is dominated by the lattice QCD determination of  $f_{D_s}$ . In semileptonic  $D$  decays, unquenched lattice QCD calculations of the  $D \rightarrow K \ell \nu$  form factor is available [13]. Using  $f_+^{DK}(0) = 0.747 \pm 0.019$  and the average of recent CLEO-c [25], Belle [26] and BABAR [42] measurements of  $B \rightarrow K \ell \nu$  decays, one obtains  $|V_{cs}| = 0.953 \pm 0.008 \pm 0.024$ , where the first error is experimental and the second, which is dominant, is from the theoretical uncertainty of the form factor. Averaging the determinations from leptonic and semileptonic decays, we find

$$|V_{cs}| = 0.986 \pm 0.016. \quad (1.10)$$

Measurements of on-shell  $W^\pm$  decays sensitive to  $|V_{cs}|$  were made by LEP-2. The  $W$  branching ratios depend on the six CKM elements involving quarks lighter than  $m_W$ . The  $W$  branching ratio to each lepton flavor is  $1/\mathcal{B}(W \rightarrow \ell \bar{\nu}_\ell) = 3[1 + \sum_{u,c,d,s,b} |V_{ij}|^2 (1 + \alpha_s(m_W)/\pi) + \dots]$ . Assuming lepton universality, the measurement  $\mathcal{B}(W \rightarrow \ell \bar{\nu}_\ell) = (10.83 \pm 0.07 \pm 0.07) \%$  [43] implies  $\sum_{u,c,d,s,b} |V_{ij}|^2 = 2.002 \pm 0.027$ . This is a precise test of unitarity; however, only flavor-tagged  $W$ -decays determine  $|V_{cs}|$  directly, such as DELPHI's tagged  $W^+ \rightarrow c \bar{s}$  analysis, yielding  $|V_{cs}| = 0.94_{-0.26}^{+0.32} \pm 0.13$  [44].

### 1.2.5. $|V_{cb}|$ :

This matrix element can be determined from exclusive and inclusive semileptonic decays of  $B$  mesons to charm. The inclusive determinations use the semileptonic decay rate measurement, together with the leptonic energy and the hadronic invariant-mass spectra. The theoretical foundation of the calculation is the operator product expansion [45,46]. It expresses the total rate and moments of differential energy and invariant-mass spectra as expansions in  $\alpha_s$ , and inverse powers of the heavy quark mass. The dependence on  $m_b$ ,  $m_c$ , and the parameters that occur at subleading order is different for different moments, and a large number of measured moments overconstrains all the parameters, and tests the consistency

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of the determination. The precise extraction of  $|V_{cb}|$  requires using a “threshold” quark mass definition [47,48]. Inclusive measurements have been performed using  $B$  mesons from  $Z^0$  decays at LEP, and at  $e^+e^-$  machines operated at the  $\Upsilon(4S)$ . At LEP, the large boost of  $B$  mesons from the  $Z^0$  allows the determination of the moments throughout phase space, which is not possible otherwise, but the large statistics available at the  $B$  factories lead to more precise determinations. An average of the measurements and a compilation of the references are provided by Ref. [14]:  $|V_{cb}| = (42.2 \pm 0.7) \times 10^{-3}$ .

Exclusive determinations are based on semileptonic  $B$  decays to  $D$  and  $D^*$ . In the  $m_{b,c} \gg \Lambda_{\text{QCD}}$  limit, all form factors are given by a single Isgur-Wise function [49], which depends on the product of the four-velocities of the  $B$  and  $D^{(*)}$  mesons,  $w = v \cdot v'$ . Heavy quark symmetry determines the normalization of the rate at  $w = 1$ , the maximum momentum transfer to the leptons, and  $|V_{cb}|$  is obtained from an extrapolation to  $w = 1$ . The exclusive determination,  $|V_{cb}| = (39.5 \pm 0.8) \times 10^{-3}$  [14], has a comparable precision to the inclusive one, and the main theoretical uncertainty in the form factor and the experimental uncertainty in the rate near  $w = 1$  are to a large extent independent of the inclusive determination. The  $V_{cb}$  and  $V_{ub}$  minireview [14] quotes a combination with a scaled error as

$$|V_{cb}| = (41.1 \pm 1.3) \times 10^{-3}. \quad (1.11)$$

Determinations of  $|V_{cb}|$  with larger uncertainties, not included in this average, can be obtained from  $\mathcal{B}(B \rightarrow D^{(*)}\tau\bar{\nu})$ . The most precise measurements involving the  $\tau$  modes are those of the  $|V_{cb}|$ -independent ratios  $\mathcal{B}(B \rightarrow D^{(*)}\tau\bar{\nu})/\mathcal{B}(B \rightarrow D^{(*)}\ell\bar{\nu})$  [50], which are 2–3 $\sigma$  above the SM predictions.

### 1.2.6. $|V_{ub}|$ :

The determination of  $|V_{ub}|$  from inclusive  $B \rightarrow X_u\ell\bar{\nu}$  decay is complicated due to large  $B \rightarrow X_c\ell\bar{\nu}$  backgrounds. In most regions of phase space where the charm background is kinematically forbidden, the hadronic physics enters via unknown nonperturbative functions, so-called shape functions. (In contrast, the nonperturbative physics for  $|V_{cb}|$  is encoded in a few parameters.) At leading order in  $\Lambda_{\text{QCD}}/m_b$ , there is only one shape function, which can be extracted from the photon energy spectrum in  $B \rightarrow X_s\gamma$  [51,52], and applied to several spectra in  $B \rightarrow X_u\ell\bar{\nu}$ . The subleading shape functions are modeled in the current determinations. Phase space cuts for which the rate has only subleading dependence on the shape function are also possible [53]. The measurements of both the hadronic and the leptonic systems are important for an optimal choice of phase space. A different approach is to make the measurements more inclusive by extending them deeper into the  $B \rightarrow X_c\ell\bar{\nu}$  region, and thus reduce the theoretical uncertainties. Analyses of the electron-energy endpoint from CLEO [54], BABAR [55], and Belle [56] quote  $B \rightarrow X_u e\bar{\nu}$  partial rates for  $|\vec{p}_e| \geq 2.0 \text{ GeV}$  and  $1.9 \text{ GeV}$ , which are well below the charm endpoint. The large and pure  $B\bar{B}$  samples at the  $B$  factories permit the selection of  $B \rightarrow X_u\ell\bar{\nu}$  decays in events where the other  $B$  is fully reconstructed [57]. With this full-reconstruction tag method, the four-momenta of both the leptonic and the hadronic systems can be

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measured. It also gives access to a wider kinematic region because of improved signal purity. Ref. 14 quotes an inclusive average as  $|V_{ub}| = (4.41 \pm 0.15^{+0.15}_{-0.19}) \times 10^{-3}$ .

To extract  $|V_{ub}|$  from an exclusive channel, the form factors have to be known. Experimentally, better signal-to-background ratios are offset by smaller yields. The  $B \rightarrow \pi \ell \bar{\nu}$  branching ratio is now known to 5%. Unquenched lattice QCD calculations of the  $B \rightarrow \pi \ell \bar{\nu}$  form factor are available [58,59] for the high  $q^2$  region ( $q^2 > 16$  or  $18 \text{ GeV}^2$ ). A fit to the experimental partial rates and lattice results versus  $q^2$  yields  $|V_{ub}| = (3.23 \pm 0.31) \times 10^{-3}$  [59]. Light-cone QCD sum rules are supposed to be applicable for  $q^2 < 12 \text{ GeV}^2$  [60]. The minireview quotes a combination,  $|V_{ub}| = (3.28 \pm 0.29) \times 10^{-3}$ .

The uncertainties in extracting  $|V_{ub}|$  from inclusive and exclusive decays are different to a large extent. A combination of the determinations is quoted by Ref. [14] as

$$|V_{ub}| = (4.13 \pm 0.49) \times 10^{-3}. \quad (1.12)$$

A determination of  $|V_{ub}|$  not included in this average can be obtained from  $\mathcal{B}(B \rightarrow \tau \bar{\nu}) = (1.14 \pm 0.22) \times 10^{-4}$  [65]. Using  $f_B = (190.5 \pm 4.2) \text{ MeV}$  [13] and  $\tau_{B^\pm} = (1.641 \pm 0.008) \text{ ps}$  [66], we find  $|V_{ub}| = (4.22 \pm 0.42) \times 10^{-3}$ . This decay is sensitive, for example, to tree-level charged Higgs contributions, and the measured rate is somewhat higher than the SM fit value.

### 1.2.7. $|V_{td}|$ and $|V_{ts}|$ :

The CKM elements  $|V_{td}|$  and  $|V_{ts}|$  are not likely to be precisely measurable in tree-level processes involving top quarks, so one has to rely on determinations from  $B-\bar{B}$  oscillations mediated by box diagrams with top quarks, or loop-mediated rare  $K$  and  $B$  decays. Theoretical uncertainties in hadronic effects limit the accuracy of the current determinations. These can be reduced by taking ratios of processes that are equal in the flavor  $SU(3)$  limit to determine  $|V_{td}/V_{ts}|$ .

The mixing of the two  $B^0$  mesons was discovered by ARGUS [61], and the mass difference is precisely measured by now,  $\Delta m_d = (0.510 \pm 0.003) \text{ ps}^{-1}$  [62]. In the  $B_s^0$  system,  $\Delta m_s$  was first measured significantly by CDF [63] and the world average, dominated by a recent LHCb measurement [64], is  $\Delta m_s = (17.761 \pm 0.022) \text{ ps}^{-1}$  [62]. Assuming  $|V_{tb}| = 1$ , and using the unquenched lattice QCD calculations  $f_{B_d} \sqrt{\widehat{B}_{B_d}} = (216 \pm 15) \text{ MeV}$  and  $f_{B_s} \sqrt{\widehat{B}_{B_s}} = (266 \pm 18) \text{ MeV}$  [13],

$$|V_{td}| = (8.4 \pm 0.6) \times 10^{-3}, \quad |V_{ts}| = (40.0 \pm 2.7) \times 10^{-3}. \quad (1.13)$$

The uncertainties are dominated by lattice QCD. Several uncertainties are reduced in the calculation of the ratio  $\xi = (f_{B_s} \sqrt{\widehat{B}_{B_s}}) / (f_{B_d} \sqrt{\widehat{B}_{B_d}}) = 1.268 \pm 0.063$  [13] and therefore the constraint on  $|V_{td}/V_{ts}|$  from  $\Delta m_d/\Delta m_s$  is more reliable theoretically. These provide a theoretically clean and significantly improved constraint

$$|V_{td}/V_{ts}| = 0.216 \pm 0.001 \pm 0.011. \quad (1.14)$$

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The inclusive branching ratio  $\mathcal{B}(B \rightarrow X_s \gamma) = (3.43 \pm 0.22) \times 10^{-4}$  extrapolated to  $E_\gamma > E_0 = 1.6 \text{ GeV}$  [67] is also sensitive to  $|V_{tb} V_{ts}|$ . In addition to  $t$ -quark penguins, a large part of the sensitivity comes from charm contributions proportional to  $V_{cb} V_{cs}^*$  via the application of  $3 \times 3$  CKM unitarity (which is used here). With the NNLO calculation of  $\mathcal{B}(B \rightarrow X_s \gamma)_{E_\gamma > E_0} / \mathcal{B}(B \rightarrow X_c e \bar{\nu})$  [68], we obtain  $|V_{ts} / V_{cb}| = 1.02 \pm 0.05$ . The  $B_s \rightarrow \mu^+ \mu^-$  rate [69] is also proportional to  $|V_{tb} V_{ts}|^2$  in the SM, and the observed signal  $\mathcal{B}(B_s \rightarrow \mu^+ \mu^-) = (2.9 \pm 0.7) \times 10^{-9}$  is consistent with the SM, with sizable uncertainties.

A complementary determination of  $|V_{td} / V_{ts}|$  is possible from the ratio of  $B \rightarrow \rho \gamma$  and  $K^* \gamma$  rates. The ratio of the neutral modes is theoretically cleaner than that of the charged ones, because the poorly known spectator-interaction contribution is expected to be smaller ( $W$ -exchange vs. weak annihilation). For now, because of low statistics we average the charged and neutral rates assuming the isospin symmetry and heavy quark limit motivated relation,  $|V_{td} / V_{ts}|^2 / \xi_\gamma^2 = [\Gamma(B^+ \rightarrow \rho^+ \gamma) + 2\Gamma(B^0 \rightarrow \rho^0 \gamma)] / [\Gamma(B^+ \rightarrow K^{*+} \gamma) + \Gamma(B^0 \rightarrow K^{*0} \gamma)] = (3.19 \pm 0.46)\%$  [67]. Here  $\xi_\gamma$  contains the poorly known hadronic physics. Using  $\xi_\gamma = 1.2 \pm 0.2$  [70], and combining the experimental and theoretical errors in quadrature, gives  $|V_{td} / V_{ts}| = 0.21 \pm 0.04$ .

A theoretically clean determination of  $|V_{td} V_{ts}^*|$  is possible from  $K^+ \rightarrow \pi^+ \nu \bar{\nu}$  decay [71]. Experimentally, only seven events have been observed [72] and the rate is consistent with the SM with large uncertainties. Much more data are needed for a precision measurement.

### 1.2.8. $|V_{tb}|$ :

The determination of  $|V_{tb}|$  from top decays uses the ratio of branching fractions  $R = \mathcal{B}(t \rightarrow Wb) / \mathcal{B}(t \rightarrow Wq) = |V_{tb}|^2 / (\sum_q |V_{tq}|^2) = |V_{tb}|^2$ , where  $q = b, s, d$ . The CDF and DØ measurements performed on data collected during Run II of the Tevatron give  $|V_{tb}| > 0.78$  [73] and  $0.99 > |V_{tb}| > 0.90$  [74], respectively, at 95% CL. CMS recently measured the same quantity at 7 TeV and gives  $|V_{tb}| > 0.92$  [75] at 95% CL.

The direct determination of  $|V_{tb}|$ , without assuming unitarity, is possible from the single top-quark-production cross section. The  $(3.51_{-0.37}^{+0.40})$  pb average cross section measured by DØ [76] and CDF [77] implies  $|V_{tb}| = 1.03 \pm 0.06$ . The average  $t$ -channel single-top cross section at CMS [78] and ATLAS [79] at 7 TeV,  $(68.5 \pm 5.8)$  pb, implies  $|V_{tb}| = 1.03 \pm 0.05$ ; the average cross section at 8 TeV from a subset of the data,  $(85 \pm 12)$  pb [80], implies  $|V_{tb}| = 0.99 \pm 0.07$ . The average of these gives

$$|V_{tb}| = 1.021 \pm 0.032. \quad (1.15)$$

This does not include correlations between the 7 and 8 TeV measurements. The experimental uncertainties dominate, and a dedicated combination would be welcome.

A weak constraint on  $|V_{tb}|$  can be obtained from precision electroweak data, where top quarks enter in loops. The sensitivity is best in  $\Gamma(Z \rightarrow b\bar{b})$  and yields  $|V_{tb}| = 0.77_{-0.24}^{+0.18}$  [81].

### 1.3. Phases of CKM elements

As can be seen from Fig. 1.1, the angles of the unitarity triangle are

$$\begin{aligned}\beta &= \phi_1 = \arg\left(-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right), \\ \alpha &= \phi_2 = \arg\left(-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right), \\ \gamma &= \phi_3 = \arg\left(-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right).\end{aligned}\tag{1.16}$$

Since  $CP$  violation involves phases of CKM elements, many measurements of  $CP$ -violating observables can be used to constrain these angles and the  $\bar{\rho}, \bar{\eta}$  parameters.

#### 1.3.1. $\epsilon$ and $\epsilon'$ :

The measurement of  $CP$  violation in  $K^0-\bar{K}^0$  mixing,  $|\epsilon| = (2.233 \pm 0.015) \times 10^{-3}$  [82], provides important information about the CKM matrix. In the SM, in the basis where  $V_{ud}V_{us}^*$  is real [83]

$$\begin{aligned}|\epsilon| &= \frac{G_F^2 f_K^2 m_K m_W^2}{12\sqrt{2}\pi^2 \Delta m_K} \widehat{B}_K \left\{ \eta_1 S(x_c) \text{Im} \left[ (V_{cs}V_{cd}^*)^2 \right] \right. \\ &\quad \left. + \eta_2 S(x_t) \text{Im} \left[ (V_{ts}V_{td}^*)^2 \right] + 2\eta_3 S(x_c, x_t) \text{Im} (V_{cs}V_{cd}^*V_{ts}V_{td}^*) \right\}\end{aligned}\tag{1.17}$$

where  $S$  is an Inami-Lim function [84],  $x_q = m_q^2/m_W^2$ , and  $\eta_i$  are perturbative QCD corrections. The constraint from  $\epsilon$  in the  $\bar{\rho}, \bar{\eta}$  plane is bounded by approximate hyperbolas. The dominant uncertainties are due to the bag parameter, for which we use  $\widehat{B}_K = 0.766 \pm 0.010$  from lattice QCD [13], and the parametric uncertainty proportional to  $\sigma(A^4)$  from  $(V_{ts}V_{td}^*)^2$ , which is approximately  $\sigma(|V_{cb}|^4)$ .

The measurement of  $6 \text{Re}(\epsilon'/\epsilon) = 1 - |\eta_{00}/\eta_{+-}|^2$ , where  $\eta_{00}$  and  $\eta_{+-}$  are the  $CP$ -violating amplitude ratios of  $K_S^0$  and  $K_L^0$  decays to two pions, provides a qualitative test of the CKM mechanism. Its nonzero experimental average,  $\text{Re}(\epsilon'/\epsilon) = (1.67 \pm 0.23) \times 10^{-3}$  [82], demonstrates the existence of direct  $CP$  violation, a prediction of the KM ansatz. While  $\text{Re}(\epsilon'/\epsilon) \propto \text{Im}(V_{td}V_{ts}^*)$ , this quantity cannot easily be used to extract CKM parameters, because the electromagnetic penguin contributions tend to cancel the gluonic penguins for large  $m_t$  [85], thereby significantly increasing the hadronic uncertainties. Most estimates [86–89] agree with the observed value, indicating that  $\bar{\eta}$  is positive. Progress in lattice QCD, in particular finite-volume calculations [90,91], may eventually provide a determination of the  $K \rightarrow \pi\pi$  matrix elements.

#### 1.3.2. $\beta / \phi_1$ :

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### 1.3.2.1. Charmonium modes:

$CP$ -violation measurements in  $B$ -meson decays provide direct information on the angles of the unitarity triangle, shown in Fig. 1.1. These overconstraining measurements serve to improve the determination of the CKM elements, or to reveal effects beyond the SM.

The time-dependent  $CP$  asymmetry of neutral  $B$  decays to a final state  $f$  common to  $B^0$  and  $\bar{B}^0$  is given by [92,93]

$$\mathcal{A}_f = \frac{\Gamma(\bar{B}^0(t) \rightarrow f) - \Gamma(B^0(t) \rightarrow f)}{\Gamma(\bar{B}^0(t) \rightarrow f) + \Gamma(B^0(t) \rightarrow f)} = S_f \sin(\Delta m_d t) - C_f \cos(\Delta m_d t), \quad (1.18)$$

where

$$S_f = \frac{2 \operatorname{Im} \lambda_f}{1 + |\lambda_f|^2}, \quad C_f = \frac{1 - |\lambda_f|^2}{1 + |\lambda_f|^2}, \quad \lambda_f = \frac{q}{p} \frac{\bar{A}_f}{A_f}. \quad (1.19)$$

Here,  $q/p$  describes  $B^0$ - $\bar{B}^0$  mixing and, to a good approximation in the SM,  $q/p = V_{tb}^* V_{td} / V_{tb} V_{td}^* = e^{-2i\beta + \mathcal{O}(\lambda^4)}$  in the usual phase convention.  $A_f$  ( $\bar{A}_f$ ) is the amplitude of the  $B^0 \rightarrow f$  ( $\bar{B}^0 \rightarrow f$ ) decay. If  $f$  is a  $CP$  eigenstate, and amplitudes with one CKM phase dominate the decay, then  $|A_f| = |\bar{A}_f|$ ,  $C_f = 0$ , and  $S_f = \sin(\arg \lambda_f) = \eta_f \sin 2\phi$ , where  $\eta_f$  is the  $CP$  eigenvalue of  $f$  and  $2\phi$  is the phase difference between the  $B^0 \rightarrow f$  and  $\bar{B}^0 \rightarrow f$  decay paths. A contribution of another amplitude to the decay with a different CKM phase makes the value of  $S_f$  sensitive to relative strong interaction phases between the decay amplitudes (it also makes  $C_f \neq 0$  possible).

The  $b \rightarrow c\bar{c}s$  decays to  $CP$  eigenstates ( $B^0 \rightarrow$  charmonium  $K_{S,L}^0$ ) are the theoretically cleanest examples, measuring  $S_f = -\eta_f \sin 2\beta$ . The  $b \rightarrow sq\bar{q}$  penguin amplitudes have dominantly the same weak phase as the  $b \rightarrow c\bar{c}s$  tree amplitude. Since only  $\lambda^2$ -suppressed penguin amplitudes introduce a new  $CP$ -violating phase, amplitudes with a single weak phase dominate, and we expect  $|\bar{A}_{\psi K} / A_{\psi K} - 1| < 0.01$ . The  $e^+e^-$  asymmetric-energy  $B$ -factory experiments, *BABAR* [95] and *Belle* [96], provide precise measurements. The world average including LHCb [97] and other measurements is [98]

$$\sin 2\beta = 0.682 \pm 0.019. \quad (1.20)$$

This measurement has a four-fold ambiguity in  $\beta$ , which can be resolved by a global fit as mentioned in Sec. 1.4. Experimentally, the two-fold ambiguity  $\beta \rightarrow \pi/2 - \beta$  (but not  $\beta \rightarrow \pi + \beta$ ) can be resolved by a time-dependent angular analysis of  $B^0 \rightarrow J/\psi K^{*0}$  [99,100], or a time-dependent Dalitz plot analysis of  $B^0 \rightarrow \bar{D}^0 h^0$  ( $h^0 = \pi^0, \eta, \omega$ ) with  $\bar{D}^0 \rightarrow K_S^0 \pi^+ \pi^-$  [101,102]. These results indicate that negative  $\cos 2\beta$  solutions are very unlikely, in agreement with the global CKM fit result.

The  $b \rightarrow c\bar{c}d$  mediated transitions, such as  $B^0 \rightarrow J/\psi \pi^0$  and  $B^0 \rightarrow D^{(*)+} D^{(*)-}$ , also measure approximately  $\sin 2\beta$ . However, the dominant component of the  $b \rightarrow d$  penguin amplitude has a different CKM phase ( $V_{tb}^* V_{td}$ ) than the tree amplitude ( $V_{cb}^* V_{cd}$ ), and

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its magnitudes are of the same order in  $\lambda$ . Therefore, the effect of penguins could be large, resulting in  $S_f \neq -\eta_f \sin 2\beta$  and  $C_f \neq 0$ . These decay modes have also been measured by *BABAR* and *Belle*. The world averages [98],  $S_{J/\psi\pi^0} = -0.93 \pm 0.15$ ,  $S_{D^+D^-} = -0.98 \pm 0.17$ , and  $S_{D^{*+}D^{*-}} = -0.71 \pm 0.09$  ( $\eta_f = +1$  for these modes), are consistent with  $\sin 2\beta$  obtained from  $B^0 \rightarrow$  charmonium  $K^0$  decays, and the  $C_f$ 's are consistent with zero, although the uncertainties are sizable.

The  $b \rightarrow c\bar{u}d$  decays,  $B^0 \rightarrow \bar{D}^0 h^0$  with  $\bar{D}^0 \rightarrow CP$  eigenstates, have no penguin contributions and provide theoretically clean  $\sin 2\beta$  measurements. *BABAR* measured  $S_{D^{(*)}h^0} = -0.56 \pm 0.25$  [94].

### 1.3.2.2. Penguin-dominated modes:

The  $b \rightarrow s\bar{q}q$  penguin-dominated decays have the same CKM phase as the  $b \rightarrow c\bar{c}s$  tree level decays, up to corrections suppressed by  $\lambda^2$ , since  $V_{tb}^*V_{ts} = -V_{cb}^*V_{cs}[1 + \mathcal{O}(\lambda^2)]$ . Therefore, decays such as  $B^0 \rightarrow \phi K^0$  and  $\eta' K^0$  provide  $\sin 2\beta$  measurements in the SM. Any new physics contribution to the amplitude with a different weak phase would give rise to  $S_f \neq -\eta_f \sin 2\beta$ , and possibly  $C_f \neq 0$ . Therefore, the main interest in these modes is not simply to measure  $\sin 2\beta$ , but to search for new physics. Measurements of many other decay modes in this category, such as  $B \rightarrow \pi^0 K_S^0$ ,  $K_S^0 K_S^0 K_S^0$ , *etc.*, have also been performed by *BABAR* and *Belle*. The results and their uncertainties are summarized in Fig. 12.3 and Table 12.1 of Ref. 93.

### 1.3.3. $\alpha / \phi_2$ :

Since  $\alpha$  is the phase between  $V_{tb}^*V_{td}$  and  $V_{ub}^*V_{ud}$ , only time-dependent  $CP$  asymmetries in  $b \rightarrow u\bar{u}d$  decay dominated modes can directly measure  $\sin 2\alpha$ , in contrast to  $\sin 2\beta$ , where several different transitions can be used. Since  $b \rightarrow d$  penguin amplitudes have a different CKM phase than  $b \rightarrow u\bar{u}d$  tree amplitudes, and their magnitudes are of the same order in  $\lambda$ , the penguin contribution can be sizable, which makes the determination of  $\alpha$  complicated. To date,  $\alpha$  has been measured in  $B \rightarrow \pi\pi$ ,  $\rho\pi$  and  $\rho\rho$  decay modes.

#### 1.3.3.1. $B \rightarrow \pi\pi$ :

It is now experimentally well established that there is a sizable contribution of  $b \rightarrow d$  penguin amplitudes in  $B \rightarrow \pi\pi$  decays. Thus,  $S_{\pi^+\pi^-}$  in the time-dependent  $B^0 \rightarrow \pi^+\pi^-$  analysis does not measure  $\sin 2\alpha$ , but

$$S_{\pi^+\pi^-} = \sqrt{1 - C_{\pi^+\pi^-}^2} \sin(2\alpha + 2\Delta\alpha), \quad (1.21)$$

where  $2\Delta\alpha$  is the phase difference between  $e^{2i\gamma}\bar{A}_{\pi^+\pi^-}$  and  $A_{\pi^+\pi^-}$ . The value of  $\Delta\alpha$ , hence  $\alpha$ , can be extracted using the isospin relation among the amplitudes of  $B^0 \rightarrow \pi^+\pi^-$ ,  $B^0 \rightarrow \pi^0\pi^0$ , and  $B^+ \rightarrow \pi^+\pi^0$  decays [103],

$$\frac{1}{\sqrt{2}} A_{\pi^+\pi^-} + A_{\pi^0\pi^0} - A_{\pi^+\pi^0} = 0, \quad (1.22)$$

and a similar expression for the  $\bar{A}_{\pi\pi}$ 's. This method utilizes the fact that a pair of pions from  $B \rightarrow \pi\pi$  decay must be in a zero angular momentum state, and, because of Bose statistics, they must have even

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isospin. Consequently,  $\pi^0\pi^\pm$  is in a pure isospin-2 state, while the penguin amplitudes only contribute to the isospin-0 final state. The latter does not hold for the electroweak penguin amplitudes, but their effect is expected to be small. The isospin analysis uses the world averages [98,104]  $S_{\pi^+\pi^-} = -0.66 \pm 0.06$ ,  $C_{\pi^+\pi^-} = -0.31 \pm 0.05$ , the branching fractions of all three modes, and the direct  $CP$  asymmetry  $C_{\pi^0\pi^0} = -0.43_{-0.24}^{+0.25}$ . This analysis leads to 16 mirror solutions for  $0 \leq \alpha < 2\pi$ . Because of this, and the sizable experimental error of the  $B^0 \rightarrow \pi^0\pi^0$  rate and  $CP$  asymmetry, only a loose constraint on  $\alpha$  can be obtained at present [105],  $0^\circ < \alpha < 10.1^\circ$ ,  $80.0^\circ < \alpha < 104.0^\circ$ ,  $119.0^\circ < \alpha < 151.0^\circ$ , and  $165.2^\circ < \alpha < 180^\circ$  at 68% CL.

### 1.3.3.2. $B \rightarrow \rho\rho$ :

The decay  $B^0 \rightarrow \rho^+\rho^-$  contains two vector mesons in the final state, which in general is a mixture of  $CP$ -even and  $CP$ -odd components. Therefore, it was thought that extracting  $\alpha$  from this mode would be complicated.

However, the longitudinal polarization fractions ( $f_L$ ) in  $B^+ \rightarrow \rho^+\rho^0$  and  $B^0 \rightarrow \rho^+\rho^-$  decays were measured to be close to unity [106], which implies that the final states are almost purely  $CP$ -even. Furthermore,  $\mathcal{B}(B^0 \rightarrow \rho^0\rho^0) = (0.97 \pm 0.24) \times 10^{-6}$  is much smaller than  $\mathcal{B}(B^0 \rightarrow \rho^+\rho^-) = (24.2_{-3.2}^{+3.1}) \times 10^{-6}$  and  $\mathcal{B}(B^+ \rightarrow \rho^+\rho^0) = (24.0_{-2.0}^{+1.9}) \times 10^{-6}$  [65], which implies that the effect of the penguin diagrams is small. The isospin analysis using the world averages,  $S_{\rho^+\rho^-} = -0.05 \pm 0.17$  and  $C_{\rho^+\rho^-} = -0.06 \pm 0.13$  [65], together with the time-dependent  $CP$  asymmetry,  $S_{\rho^0\rho^0} = -0.3 \pm 0.7$  and  $C_{\rho^0\rho^0} = -0.2 \pm 0.9$  [107], and the above-mentioned branching fractions, gives  $0^\circ < \alpha < 5.4^\circ$ ,  $84.6^\circ < \alpha < 95.3^\circ$  and  $174.8^\circ < \alpha < 180^\circ$  at 68% CL [105], with mirror solutions at  $3\pi/2 - \alpha$ . A possible small violation of Eq. (1.22) due to the finite width of the  $\rho$  [108] is neglected.

### 1.3.3.3. $B \rightarrow \rho\pi$ :

The final state in  $B^0 \rightarrow \rho^+\pi^-$  decay is not a  $CP$  eigenstate, but this decay proceeds via the same quark-level diagrams as  $B^0 \rightarrow \pi^+\pi^-$ , and both  $B^0$  and  $\bar{B}^0$  can decay to  $\rho^+\pi^-$ . Consequently, mixing-induced  $CP$  violations can occur in four decay amplitudes,  $B^0 \rightarrow \rho^\pm\pi^\mp$  and  $\bar{B}^0 \rightarrow \rho^\pm\pi^\mp$ . The time-dependent Dalitz plot analysis of  $B^0 \rightarrow \pi^+\pi^-\pi^0$  decays permits the extraction of  $\alpha$  with a single discrete ambiguity,  $\alpha \rightarrow \alpha + \pi$ , since one knows the variation of the strong phases in the interference regions of the  $\rho^+\pi^-$ ,  $\rho^-\pi^+$ , and  $\rho^0\pi^0$  amplitudes in the Dalitz plot [109]. The combination of Belle [110] and BABAR [111] measurements gives  $\alpha = (54.1_{-10.3}^{+7.7})^\circ$  and  $(141.8_{-5.4}^{+4.7})^\circ$  [105]. This constraint is still moderate.

Combining the  $B \rightarrow \pi\pi$ ,  $\rho\pi$ , and  $\rho\rho$  decay modes [105],  $\alpha$  is constrained as

$$\alpha = (85.4_{-3.8}^{+3.9})^\circ. \quad (1.23)$$

A different statistical approach [112] gives similar constraint from the combination of these measurements.

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### 1.3.4. $\gamma / \phi_3$ :

By virtue of Eq. (1.16),  $\gamma$  does not depend on CKM elements involving the top quark, so it can be measured in tree-level  $B$  decays. This is an important distinction from the measurements of  $\alpha$  and  $\beta$ , and implies that the measurements of  $\gamma$  are unlikely to be affected by physics beyond the SM.

#### 1.3.4.1. $B^\pm \rightarrow DK^\pm$ :

The interference of  $B^- \rightarrow D^0 K^-$  ( $b \rightarrow c\bar{u}s$ ) and  $B^- \rightarrow \bar{D}^0 K^-$  ( $b \rightarrow u\bar{c}s$ ) transitions can be studied in final states accessible in both  $D^0$  and  $\bar{D}^0$  decays [92]. In principle, it is possible to extract the  $B$  and  $D$  decay amplitudes, the relative strong phases, and the weak phase  $\gamma$  from the data.

A practical complication is that the precision depends sensitively on the ratio of the interfering amplitudes

$$r_B = \left| A(B^- \rightarrow \bar{D}^0 K^-) / A(B^- \rightarrow D^0 K^-) \right|, \quad (1.24)$$

which is around 0.1–0.2. The original GLW method [113,114] considers  $D$  decays to  $CP$  eigenstates, such as  $B^\pm \rightarrow D_{CP}^{(*)}(\rightarrow \pi^+\pi^-)K^\pm$ . To alleviate the smallness of  $r_B$  and make the interfering amplitudes (which are products of the  $B$  and  $D$  decay amplitudes) comparable in magnitude, the ADS method [115] considers final states where Cabibbo-allowed  $\bar{D}^0$  and doubly-Cabibbo-suppressed  $D^0$  decays interfere. Extensive measurements have been made by the  $B$  factories [116,117], CDF [118] and LHCb [119] using both methods.

It was realized that both  $D^0$  and  $\bar{D}^0$  have large branching fractions to certain three-body final states, such as  $K_S\pi^+\pi^-$ , and the analysis can be optimized by studying the Dalitz plot dependence of the interferences [120,121]. The best present determination of  $\gamma$  comes from this method. Belle [122] and BABAR [123] obtained  $\gamma = (78_{-12}^{+11} \pm 4 \pm 9)^\circ$  and  $\gamma = (68 \pm 14 \pm 4 \pm 3)^\circ$ , respectively, where the last uncertainty is due to the  $D$ -decay modeling. (LHCb also measured  $\gamma = (44_{-38}^{+43})^\circ$  with the same method [124].) The error is sensitive to the central value of the amplitude ratio  $r_B$  (and  $r_B^*$  for the  $D^*K$  mode), for which Belle found somewhat larger central values than BABAR. The same values of  $r_B^{(*)}$  enter the ADS analyses, and the data can be combined to fit for  $r_B^{(*)}$  and  $\gamma$ . The  $D^0$ – $\bar{D}^0$  mixing has been neglected in all measurements, but its effect on  $\gamma$  is far below the present experimental accuracy [125], unless  $D^0$ – $\bar{D}^0$  mixing is due to  $CP$ -violating new physics, in which case it can be included in the analysis [126].

Combining the GLW, ADS, and Dalitz analyses [105],  $\gamma$  is constrained as

$$\gamma = (68.0_{-8.5}^{+8.0})^\circ. \quad (1.25)$$

Similar results are found in Ref. [112].

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### 1.3.4.2. $B^0 \rightarrow D^{(*)\pm}\pi^\mp$ :

The interference of  $b \rightarrow u$  and  $b \rightarrow c$  transitions can be studied in  $\bar{B}^0 \rightarrow D^{(*)+}\pi^-$  ( $b \rightarrow c\bar{u}d$ ) and  $\bar{B}^0 \rightarrow B^0 \rightarrow D^{(*)+}\pi^-$  ( $\bar{b} \rightarrow \bar{u}c\bar{d}$ ) decays and their  $CP$  conjugates, since both  $B^0$  and  $\bar{B}^0$  decay to  $D^{(*)\pm}\pi^\mp$  (or  $D^\pm\rho^\mp$ , etc.). Since there are only tree and no penguin contributions to these decays, in principle, it is possible to extract from the four time-dependent rates the magnitudes of the two hadronic amplitudes, their relative strong phase, and the weak phase between the two decay paths, which is  $2\beta + \gamma$ .

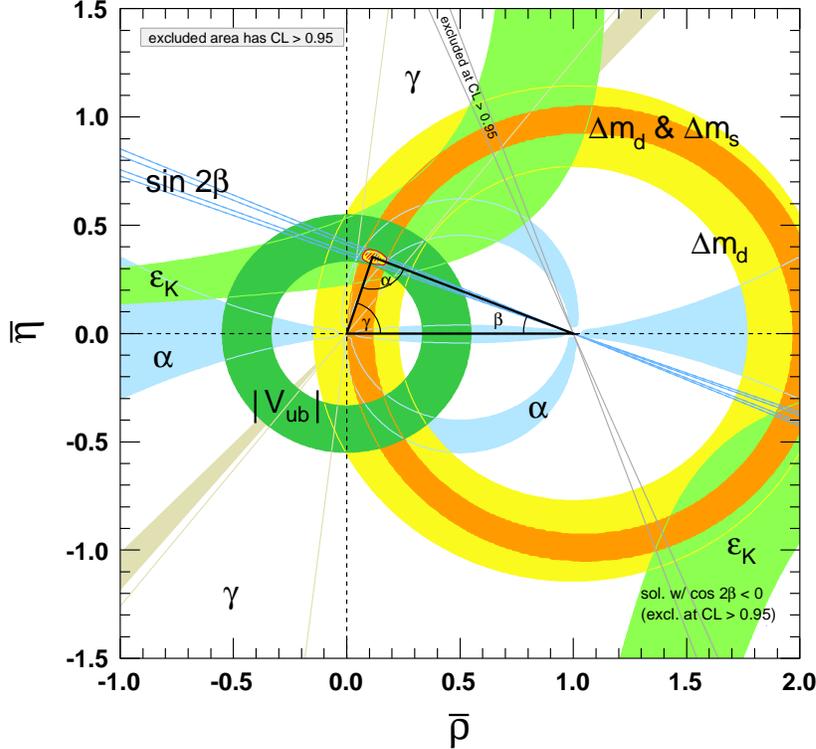
A complication is that the ratio of the interfering amplitudes is very small,  $r_{D\pi} = A(B^0 \rightarrow D^+\pi^-)/A(\bar{B}^0 \rightarrow D^+\pi^-) = \mathcal{O}(0.01)$  (and similarly for  $r_{D^*\pi}$  and  $r_{D\rho}$ ), and therefore it has not been possible to measure it. To obtain  $2\beta + \gamma$ ,  $SU(3)$  flavor symmetry and dynamical assumptions have been used to relate  $A(\bar{B}^0 \rightarrow D^-\pi^+)$  to  $A(\bar{B}^0 \rightarrow D_s^-\pi^+)$ , so this measurement is not model-independent at present. Combining the  $D^\pm\pi^\mp$ ,  $D^{*\pm}\pi^\mp$  and  $D^\pm\rho^\mp$  measurements [127] gives  $\sin(2\beta + \gamma) > 0.68$  at 68% CL [105], consistent with the previously discussed results for  $\beta$  and  $\gamma$ . The amplitude ratio is much larger in the analogous  $B_s^0 \rightarrow D_s^\pm K^\mp$  decays, which will allow a model-independently extraction of  $\gamma - 2\beta_s$  [128] at LHCb [129] (here  $\beta_s = \arg(-V_{ts}V_{tb}^*/V_{cs}V_{cb}^*)$  is related to the phase of  $B_s$  mixing).

## 1.4. Global fit in the Standard Model

Using the independently measured CKM elements mentioned in the previous sections, the unitarity of the CKM matrix can be checked. We obtain  $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9999 \pm 0.0006$  (1st row),  $|V_{cd}|^2 + |V_{cs}|^2 + |V_{cb}|^2 = 1.024 \pm 0.032$  (2nd row),  $|V_{ud}|^2 + |V_{cd}|^2 + |V_{td}|^2 = 1.000 \pm 0.004$  (1st column), and  $|V_{us}|^2 + |V_{cs}|^2 + |V_{ts}|^2 = 1.025 \pm 0.032$  (2nd column), respectively. The uncertainties in the second row and column are dominated by that of  $|V_{cs}|$ . For the second row, a slightly better check is obtained from the measurement of  $\sum_{u,c,d,s,b} |V_{ij}|^2$  in Sec. 1.2.4 minus the sum in the first row above:  $|V_{cd}|^2 + |V_{cs}|^2 + |V_{cb}|^2 = 1.002 \pm 0.027$ . These provide strong tests of the unitarity of the CKM matrix. With the significantly improved direct determination of  $|V_{tb}|$ , the unitarity checks for the third row and column have also become fairly precise, leaving decreasing room for mixing with other states. The sum of the three angles of the unitarity triangle,  $\alpha + \beta + \gamma = (175 \pm 9)^\circ$ , is also consistent with the SM expectation.

The CKM matrix elements can be most precisely determined using a global fit to all available measurements and imposing the SM constraints (*i.e.*, three generation unitarity). The fit must also use theory predictions for hadronic matrix elements, which sometimes have significant uncertainties. There are several approaches to combining the experimental data. CKMfitter [6,105] and Ref. 130 (which develops [131,132] further) use frequentist statistics, while UTfit [112,133] uses a Bayesian approach. These approaches provide similar results.

The constraints implied by the unitarity of the three generation CKM matrix significantly reduce the allowed range of some of the



**Figure 1.2:** Constraints on the  $\bar{\rho}, \bar{\eta}$  plane. The shaded areas have 95% CL.

CKM elements. The fit for the Wolfenstein parameters defined in Eq. (1.4) gives

$$\begin{aligned} \lambda &= 0.22537 \pm 0.00061, & A &= 0.814^{+0.023}_{-0.024}, \\ \bar{\rho} &= 0.117 \pm 0.021, & \bar{\eta} &= 0.353 \pm 0.013. \end{aligned} \quad (1.26)$$

These values are obtained using the method of Refs. [6,105]. Using the prescription of Refs. [112,133] gives  $\lambda = 0.2255 \pm 0.0006$ ,  $A = 0.818 \pm 0.015$ ,  $\bar{\rho} = 0.124 \pm 0.024$ ,  $\bar{\eta} = 0.354 \pm 0.015$  [134]. The fit results for the magnitudes of all nine CKM elements are

$$V_{\text{CKM}} = \begin{pmatrix} 0.97427 \pm 0.00014 & 0.22536 \pm 0.00061 & 0.00355 \pm 0.00015 \\ 0.22522 \pm 0.00061 & 0.97343 \pm 0.00015 & 0.0414 \pm 0.0012 \\ 0.00886^{+0.00033}_{-0.00032} & 0.0405^{+0.0011}_{-0.0012} & 0.99914 \pm 0.00005 \end{pmatrix}, \quad (1.27)$$

and the Jarlskog invariant is  $J = (3.06^{+0.21}_{-0.20}) \times 10^{-5}$ .

Figure 1.2 illustrates the constraints on the  $\bar{\rho}, \bar{\eta}$  plane from various measurements and the global fit result. The shaded 95% CL regions all overlap consistently around the global fit region.

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### 1.5. Implications beyond the SM

The effects in  $B$ ,  $B_s$ ,  $K$ , and  $D$  decays and mixings due to high-scale physics ( $W$ ,  $Z$ ,  $t$ ,  $H$  in the SM, and unknown heavier particles) can be parameterized by operators composed of SM fields, obeying the  $SU(3) \times SU(2) \times U(1)$  gauge symmetry. Flavor-changing neutral currents, suppressed in the SM, are especially sensitive to beyond SM (BSM) contributions. Processes studied in great detail, both experimentally and theoretically, include neutral meson mixings,  $B_{(s)} \rightarrow X\gamma$ ,  $X\ell^+\ell^-$ ,  $\ell^+\ell^-$ ,  $K \rightarrow \pi\nu\bar{\nu}$ , etc. The BSM contributions to these operators are suppressed by powers of the scale of new physics. Already at lowest order, there are many dimension-6 operators, and the observable effects of BSM interactions are encoded in their coefficients. In the SM, these coefficients are determined by just the four CKM parameters, and the  $W$ ,  $Z$ , and quark masses. For example,  $\Delta m_d$ ,  $\Gamma(B \rightarrow \rho\gamma)$ ,  $\Gamma(B \rightarrow \pi\ell^+\ell^-)$ , and  $\Gamma(B \rightarrow \ell^+\ell^-)$  are all proportional to  $|V_{td}V_{tb}|^2$  in the SM, however, they may receive unrelated contributions from new physics. The new physics contributions may or may not obey the SM relations. (For example, the flavor sector of the MSSM contains 69  $CP$ -conserving parameters and 41  $CP$ -violating phases, *i.e.*, 40 new ones [135]). Thus, similar to the measurements of  $\sin 2\beta$  in tree- and loop-dominated decay modes, overconstraining measurements of the magnitudes and phases of flavor-changing neutral-current amplitudes give good sensitivity to new physics.

To illustrate the level of suppression required for BSM contributions, consider a class of models in which the unitarity of the CKM matrix is maintained, and the dominant effect of new physics is to modify the neutral meson mixing amplitudes [136] by  $(z_{ij}/\Lambda^2)(\bar{q}_i\gamma^\mu P_L q_j)^2$  (for recent reviews, see [137,138]). It is only known since the measurements of  $\gamma$  and  $\alpha$  that the SM gives the leading contribution to  $B^0 - \bar{B}^0$  mixing [6,139]. Nevertheless, new physics with a generic weak phase may still contribute to neutral meson mixings at a significant fraction of the SM [140,141,133]. The existing data imply that  $\Lambda/|z_{ij}|^{1/2}$  has to exceed about  $10^4$  TeV for  $K^0 - \bar{K}^0$  mixing,  $10^3$  TeV for  $D^0 - \bar{D}^0$  mixing, 500 TeV for  $B^0 - \bar{B}^0$  mixing, and 100 TeV for  $B_s^0 - \bar{B}_s^0$  mixing [133,138]. (Some other operators are even better constrained [133].) The constraints are the strongest in the kaon sector, because the CKM suppression is the most severe. Thus, if there is new physics at the TeV scale,  $|z_{ij}| \ll 1$  is required. Even if  $|z_{ij}|$  are suppressed by a loop factor and  $|V_{ti}^*V_{tj}|^2$  (in the down quark sector), similar to the SM, one expects percent-level effects, which may be observable in forthcoming flavor physics experiments. To constrain such extensions of the SM, many measurements irrelevant for the SM-CKM fit, such as the  $CP$  asymmetry in semileptonic  $B_{d,s}^0$  decays,  $A_{\text{SL}}^{d,s}$ , are important [142]. A  $D\mathcal{O}$  measurement sensitive to certain linear combinations of  $A_{\text{SL}}^d$  and  $A_{\text{SL}}^s$  shows a  $3.6\sigma$  hint of a deviation from the SM [143].

Many key measurements which are sensitive to BSM flavor physics are not useful to think about in terms of constraining the unitarity triangle in Fig. 1.1. For example, besides the angles in Eq. (1.16), a key quantity in the  $B_s$  system is  $\beta_s = \arg(-V_{ts}V_{tb}^*/V_{cs}V_{cb}^*)$ ,

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which is the small,  $\lambda^2$ -suppressed, angle of a “squashed” unitarity triangle, obtained by taking the scalar product of the second and third columns. This angle can be measured via time-dependent  $CP$  violation in  $B_s^0 \rightarrow J/\psi \phi$ , similar to  $\beta$  in  $B^0 \rightarrow J/\psi K^0$ . Since the  $J/\psi \phi$  final state is not a  $CP$  eigenstate, an angular analysis of the decay products is needed to separate the  $CP$ -even and  $CP$ -odd components, which give opposite asymmetries. In the SM, the asymmetry for the  $CP$ -even part is  $2\beta_s$  (sometimes the notation  $\phi_s = -2\beta_s$  plus a possible BSM contribution to the  $B_s$  mixing phase is used). Testing if the data agree with the SM prediction,  $2\beta_s = 0.0363 \pm 0.0018$  [105], is another sensitive test of the SM. After the first Tevatron  $CP$ -asymmetry measurements of  $B_s^0 \rightarrow J/\psi \phi$  hinted at a possible tension with the SM, the current world average, dominated by LHCb [145] and including  $B_s \rightarrow J/\psi K^+ K^-$  and  $J/\psi \pi^+ \pi^-$  measurements, is  $2\beta_s = 0.00 \pm 0.07$  [65]. This uncertainty is about 40 times the SM uncertainty; thus a lot will be learned from higher precision measurements in the future.

In the kaon sector, the two measured  $CP$ -violating observables  $\epsilon$  and  $\epsilon'$  are tiny, so models in which all sources of  $CP$  violation are small were viable before the  $B$ -factory measurements. Since the measurement of  $\sin 2\beta$ , we know that  $CP$  violation can be an  $\mathcal{O}(1)$  effect, and only flavor mixing is suppressed between the three quark generations. Thus, many models with spontaneous  $CP$  violation are excluded. In the kaon sector, a very clean test of the SM will come from measurements of  $K^+ \rightarrow \pi^+ \nu \bar{\nu}$  and  $K_L^0 \rightarrow \pi^0 \nu \bar{\nu}$ . These loop-induced rare decays are sensitive to new physics, and will allow a determination of  $\beta$ , independent of its value measured in  $B$  decays [146].

The CKM elements are fundamental parameters, so they should be measured as precisely as possible. The overconstraining measurements of  $CP$  asymmetries, mixing, semileptonic, and rare decays severely constrain the magnitudes and phases of possible new physics contributions to flavor-changing interactions. If new particles are observed at the LHC, it will be important to explore their flavor parameters as precisely as possible to understand the underlying physics.