We adjust N to obtain a desired ε. For $\mu_B = 0$ this converges to Eq. (17.32). As in that case (see the last paragraph of Section 17.5.2) this gives a *conservative* upper limit in that for any given true μ_S we get a true probability $\geq 1 - \varepsilon$ that $N \geq \mu_S$, averaged over a large set of identically performed experiments. For $\varepsilon = 0.10$, Fig. 17.8 shows N as a function of n_0 and μ_B.

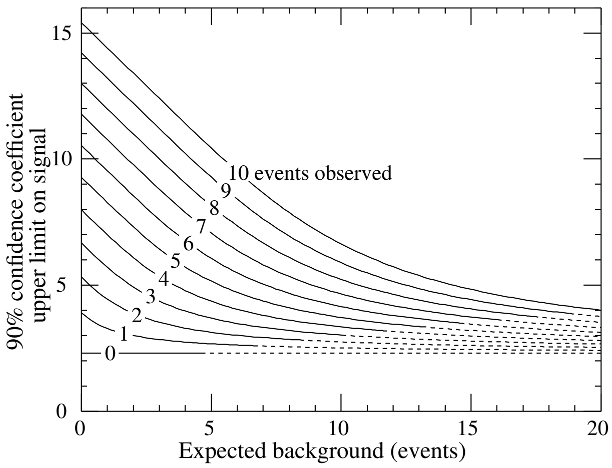

Figure 17.8: 90% confidence coefficient upper limit on the number of signal events as a function of the expected number of background events. For example, if the expected background is 8 events and 5 events are observed, then the signal is 4.0 (approximately) or less with 90% confidence. Dashed portions indicate regions where it is to be expected that the number observed would exceed the number actually observed $\geq 99\%$ of the time, even in the complete absence of signal.

Averaging of experiments and other comparisons require that n_0 and μ_B be quoted and the technique used for upper limit extraction be given.

If $\mu_B \gg n_0$ the experimenter should question the probability of observing n_B as that n_0. If this is very small the background, μ_B, may not have been calculated properly and the upper limit for μ_S obtained under those assumptions may be too low. For example, in Fig. 17.8, the dashed portions of the curves lie in the region where n_0 is expected to exceed the observed value 99% of the time (or more), even in the complete absence of signal. In these regions one should be cautious about accepting the results of the measurement.

As in the Gaussian case (Sec. 17.5.3), whenever $n_0 < \mu_B$ some experimenters may prefer to use N calculated as if $n_0 \approx \mu_B$ rather than the smaller value obtained from the observed n_0.

17.6. Propagation of errors

Suppose we have a set of N random variables y_i which may be direct measurements or derived estimators $\widehat{\alpha}$, and we have a covariance matrix $V(y)$ for these. We can make a transformation to a different set of variables $f_n \equiv f_n(y)$, $j = 1, \ldots, M$ $(M \leq N)$ and obtain best estimates for the f_n from

$$\widehat{f}_n \approx f_n(\widehat{y}) + \frac{1}{2} \sum_{k,n}^{N} V_{kn}(\widehat{y}) \left[\frac{\partial^2 f_n}{\partial y_k \, \partial y_n}\right]_{\widehat{y}} \tag{17.36}$$

with covariance matrix

$$V_{ij}(\widehat{f}) \approx \sum_{n,m} \left.\frac{\partial f_i}{\partial y_n}\right|_{\widehat{y}} \left.\frac{\partial f_j}{\partial y_m}\right|_{\widehat{y}} V_{nm}(\widehat{y}) \ . \tag{17.37}$$

For a single-valued function f of a single measurement y with variance σ^2 (*i.e.*, $M = 1, N = 1$), this becomes

$$\begin{aligned} \widehat{f} &\approx f(\widehat{y}) + \frac{1}{2}\sigma^2 f''(\widehat{y}) \\ V(\widehat{f}) &\approx \sigma^2 [f'(\widehat{y})]^2 \ , \end{aligned} \tag{17.38}$$

where the primes denote differentiation with respect to y, evaluated at $\widehat{y}$.

These approximations are based on a Taylor expansion of f about the true value of y. If f is approximately linear in y over a range of roughly $\widehat{y}_i \pm \sigma(y_i)$, the approximation is good and the second-order terms in (17.36) and (17.38) can be neglected. This is what is usually done. However, if linearity is badly violated (*e.g.*, $f \propto 1/y$ and $\widehat{y}$ is no more than a few σ from zero), it should be recognized that propagation of errors will give very approximate results. In such cases $\widehat{f} \approx f(\widehat{y})$ may be a biased estimator for f even if $\widehat{y}$ is unbiased for y, and the second-order terms in (17.36) and (17.38) will help to reduce that bias.

*In addition to the references cited, communications with R.D. Cousins, F. James, G. Lynch, and B. Roe have been invaluable in formulating this section.

†There is an additional problem: Even if we set a confidence limit on m_ν^2 by some particular recipe, it translates into a different confidence limit on $\sqrt{m_\nu^2}$ except when a Bayesian procedure with Jeffreys' $\pi(\alpha) \propto 1/\alpha$ prior distribution is used.

References:

1. H. Cramér, *Mathematical Methods of Statistics,* Princeton Univ. Press, New Jersey (1958).
2. J. Orear, "Notes on Statistics for Physicists," Univ. of Calif. Radiation Lab. Report UCRL-8417 (1958), unpublished; "Notes on Statistics for Physicists, Revised," Cornell Univ. Report CLNS 82/511 (1982), unpublished.
3. F.T. Solmitz, Annu. Rev. Nucl. Sci. **14**, 375–402 (1964).
4. W.T. Eadie, D. Drijard, F.E. James, M. Roos, and B. Sadoulet, *Statistical Methods in Experimental Physics* (North Holland, Amsterdam and London, 1971).
5. L. Lyons, *Statistics for Nuclear and Particle Physicists,* (Cambridge University Press, New York (1986)).
6. B.P. Roe, *Probability and Statistics in Experimental Physics,* (Springer-Verlag, New York, 208 pp., 1992).
7. S. Baker and R. Cousins, Nucl. Instr. and Meth. **221**, 437 (1984).
8. W.H. Press *et al.*, *Numerical Recipes* (Cambridge University Press, New York, 1986).
9. W.H. Maindonald *et al.*, *Statistical Computation* (John Wiley and Sons, Inc., New York, 1984).
10. A. Basilevsky *et al.*, *Applied Matrix Algebra in the Statistical Sciences* (North Holland, New York, 1983).
11. F. James and M. Roos, "MINUIT, Function Minimization and Error Analysis," CERN D506 (Long Writeup). Available from the CERN Program Library Office, CERN-DD Division, CERN, CH-1211, Geneva 21, Switzerland.
12. Microsoft® is a registered trademark of Microsoft Corporation.
13. M. Abramowitz and I. Stegun, eds., *Handbook of Mathematical Functions* (Dover, New York, 1972).
14. F. James and M. Roos, Phys. Rev. **D44**, 299 (1991).
15. J. Neyman, *Lectures and Conferences on Mathematical Statistics,*" Washington (1938).
16. H. Jeffreys, *Theory of Probability,* 3rd ed. (Oxford Univ. Press, Oxford, England, 447 pp., 1961).
17. G. Lynch, "Calculations of Upper Limits," Particle Data Group Note PDG-93-03 (Sept. 1993) (unpublished).
18. O. Helene, Nucl. Instr. and Meth. **212**, 319 (1983).