

More on Fitting with Mass Differences

Orin Dahl
 October 5, 1993

We wish to do fits to particle masses for the case where at least one experiment has quoted measured values for two masses (A and B) and for their difference (D). The experiment has constrained D so that

$$D = A - B \tag{1}$$

The experiment quotes $A \pm a$, $B \pm b$, and $D \pm d$. Because of systematic errors, the error d is smaller than it would be if the measurements of A and B were independent.

We would like to use all three measurements in the fit, but we must be careful. Because of the constraint (1), the three measurements are not independent; there are really only two independent measurements with some correlation between them. We may express this by

$$d^2 = a^2 + b^2 - 2abc \tag{2}$$

where c is the correlation between a and b .

Gerry Lynch has described an elegant method for using all three measurements.¹ He forms modified errors for each quantity that give the quoted errors when we do a fit that assumes that the three quantities are not correlated. In his approach, we put into the fit $A \pm x$, $B \pm y$, and $D \pm z$ and have no correlations. The equations that one must solve to obtain x , y , and z he writes as:

$$\frac{1}{a^2} = \frac{1}{x^2} + \frac{1}{y^2 + z^2}, \quad \frac{1}{b^2} = \frac{1}{y^2} + \frac{1}{x^2 + z^2}, \quad \frac{1}{d^2} = \frac{1}{z^2} + \frac{1}{x^2 + y^2} \tag{3}$$

Unfortunately, Gerry was unable to find analytic solutions to these equations.

However, a different method of deriving the uncorrelated errors x , y , and z will lead us directly to the desired solutions. Since the measurements have been constrained, only two of them are independent and we may write the contribution to χ^2 in terms of only 2 of the 3 measurements. We shall use A and B . Thus (using matrix notation)

$$\chi^2 = (\alpha \ \beta) \begin{pmatrix} a^2 & abc \\ abc & b^2 \end{pmatrix}^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{4}$$

where we define α , β , and δ to be the difference between the measurement and the final fit for A , B , and D . After some algebra, we find that

$$\chi^2 = \frac{\alpha^2 b^2 + \beta^2 a^2 - 2\alpha\beta abc}{a^2 b^2 (1 - c^2)} \tag{5}$$

¹ Gerry Lynch, Fitting with Mass Differences, Particle Data Group Note PDG–93–01

However, since both the measurement and the final fit are constrained by equation (1), we have that

$$\delta = \alpha - \beta$$

or

$$\delta^2 = \alpha^2 + \beta^2 - 2\alpha\beta \quad (6)$$

Thus, we may write χ^2 as

$$\chi^2 = \frac{\alpha^2(b^2 - abc) + \beta^2(a^2 - abc) + \delta^2 abc}{a^2 b^2 (1 - c^2)} \quad (7)$$

Now we wish to write χ^2 in terms of the modified errors x , y , and z and no correlations, we have

$$\chi^2 = \frac{\alpha^2}{x^2} + \frac{\beta^2}{y^2} + \frac{\delta^2}{z^2} \quad (8)$$

By inspection

$$\begin{aligned} \frac{1}{x^2} &= \frac{b^2 - abc}{a^2 b^2 (1 - c^2)} \\ \frac{1}{y^2} &= \frac{a^2 - abc}{a^2 b^2 (1 - c^2)} \\ \frac{1}{z^2} &= \frac{abc}{a^2 b^2 (1 - c^2)} \end{aligned} \quad (9)$$

If we define

$$q = 4a^2 b^2 (1 - c^2) \quad (10)$$

we find

$$q = 2(b^2 d^2 + a^2 d^2 + a^2 b^2) - (a^4 + b^4 + d^4) \quad (11)$$

and

$$\begin{aligned} \frac{1}{x^2} &= \frac{2(b^2 + d^2 - a^2)}{q} \\ \frac{1}{y^2} &= \frac{2(a^2 + d^2 - b^2)}{q} \\ \frac{1}{z^2} &= \frac{2(a^2 + b^2 - d^2)}{q} \end{aligned} \quad (12)$$

After a little algebra, we get

$$\begin{aligned} x^2 &= a^2 + \frac{(a^2 + b^2 - d^2)(a^2 + d^2 - b^2)}{2(b^2 + d^2 - a^2)} \\ y^2 &= b^2 + \frac{(a^2 + b^2 - d^2)(b^2 + d^2 - a^2)}{2(a^2 + d^2 - b^2)} \\ z^2 &= d^2 + \frac{(a^2 + d^2 - b^2)(b^2 + d^2 - a^2)}{2(a^2 + b^2 - d^2)} \end{aligned} \quad (13)$$

These equations may be solved for a^2 , b^2 , and d^2 to give equation (3).

Looking at equation (12), we see that the original errors must satisfy the inequalities

$$a^2 < b^2 + d^2, \quad b^2 < a^2 + d^2, \quad d^2 < a^2 + b^2. \quad (14)$$