$K^\pm_{\ell 3}$ AND $K^0_{\ell 3}$ FORM FACTORS
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Assuming that only the vector current contributes to $K \to \pi \ell \nu$ decays, we write the matrix element as

$$M \propto f_+(t) \left[ (P_K + P_\pi) \overline{\mu} \gamma_\mu (1 + \gamma_5) \nu \right] + f_-(t) \left[ m_\ell \overline{\nu} (1 + \gamma_5) \nu \right] ,$$

where $P_K$ and $P_\pi$ are the four-momenta of the $K$ and $\pi$ mesons, $m_\ell$ is the lepton mass, and $f_+$ and $f_-$ are dimensionless form factors which can depend only on $t = (P_K - P_\pi)^2$, the square of the four-momentum transfer to the leptons. If time-reversal invariance holds, $f_+$ and $f_-$ are relatively real. $K_{\mu 3}$ experiments measure $f_+$ and $f_-$, while $K_{e 3}$ experiments are sensitive only to $f_+$ because the small electron mass makes the $f_-$ term negligible.

(a) $K_{\mu 3}$ experiments. Analyses of $K_{\mu 3}$ data frequently assume a linear dependence of $f_+$ and $f_-$ on $t$, i.e.,

$$f_\pm(t) = f_\pm(0) \left[ 1 + \lambda_\pm(t/m_\pi^2) \right]$$

Most $K_{\mu 3}$ data are adequately described by Eq. (2) for $f_+$ and a constant $f_-$ (i.e., $\lambda_- = 0$). There are two equivalent parametrizations commonly used in these analyses:

(1) $\lambda_+, \xi(0)$ parametrization. Analyses of $K_{\mu 3}$ data often introduce the ratio of the two form factors

$$\xi(t) = f_-(t)/f_+(t) .$$

The $K_{\mu 3}$ decay distribution is then described by the two parameters $\lambda_+$ and $\xi(0)$ (assuming time reversal invariance and $\lambda_- = 0$). These parameters can be determined by three different methods:

**Method A.** By studying the Dalitz plot or the pion spectrum of $K_{\mu 3}$ decay. The Dalitz plot density is (see, e.g., Chounet et al. [1]):

$$\rho(E_\pi, E_\mu) \propto f_+^2(t) \left[ A + B\xi(t) + C\xi(t)^2 \right] ,$$
where

\[
A = m_K \left(2E_\mu E_\nu - m_K E'_\pi\right) + m^2_\mu \left(\frac{1}{4}E'_\pi - E_\nu\right),
\]

\[
B = m^2_\mu \left(E_\nu - \frac{1}{2}E'_\pi\right),
\]

\[
C = \frac{1}{4}m^2_\mu E'_\pi,
\]

\[
E'_\pi = E^{\text{max}}_\pi - E_\pi = (m^2_K + m^2_\pi - m^2_\mu) / 2m_K - E_\pi. \tag{4}
\]

Here \(E_\pi\), \(E_\mu\), and \(E_\nu\) are, respectively, the pion, muon, and neutrino energies in the kaon center of mass. The density \(\rho\) is fit to the data to determine the values of \(\lambda_+\), \(\xi(0)\), and their correlation.

**Method B.** By measuring the \(K_{\mu3}/K_{e3}\) branching ratio and comparing it with the theoretical ratio (see, e.g., Fearing et al. [2]) as given in terms of \(\lambda_+\) and \(\xi(0)\), assuming \(\mu-e\) universality:

\[
\Gamma(K^{\pm}_{\mu3})/\Gamma(K^{\pm}_{e3}) = 0.6457 + 1.4115\lambda_+ + 0.1264\xi(0) + 0.0192\xi(0)^2 + 0.0080\lambda_+\xi(0),
\]

\[
\Gamma(K^0_{\mu3})/\Gamma(K^0_{e3}) = 0.6452 + 1.3162\lambda_+ + 0.1264\xi(0) + 0.0186\xi(0)^2 + 0.0064\lambda_+\xi(0). \tag{5}
\]

This cannot determine \(\lambda_+\) and \(\xi(0)\) simultaneously but simply fixes a relationship between them.

**Method C.** By measuring the muon polarization in \(K_{\mu3}\) decay. In the rest frame of the \(K\), the \(\mu\) is expected to be polarized in the direction \(A\) with \(P = A/|A|\), where \(A\) is given (Cabibbo and Maksymowicz [3]) by

\[
A = a_1(\xi)p_\mu
\]

\[
- a_2(\xi) \left[ \frac{p_\mu}{m_\mu} \left( m_K - E_\pi + \frac{p_\pi \cdot p_\mu (E_\mu - m_\mu)}{|p_\mu|^2} \right) + p_\pi \right]
\]

\[
+ m_K \text{Im} \xi(t)(p_\pi \times p_\mu). \tag{6}
\]
If time-reversal invariance holds, $\xi$ is real, and thus there is no polarization perpendicular to the $K$-decay plane. Polarization experiments measure the weighted average of $\xi(t)$ over the $t$ range of the experiment, where the weighting accounts for the variation with $t$ of the sensitivity to $\xi(t)$.

(2) $\lambda_+, \lambda_0$ parametrization. Most of the more recent $K_{\mu 3}$ analyses have parameterized in terms of the form factors $f_+$ and $f_0$ which are associated with vector and scalar exchange, respectively, to the lepton pair. $f_0$ is related to $f_+$ and $f_-$ by

$$f_0(t) = f_+(t) + \left[ t/(m_K^2 - m_\pi^2) \right] f_-(t).$$

(7)

Here $f_0(0)$ must equal $f_+(0)$ unless $f_-(t)$ diverges at $t = 0$. The earlier assumption that $f_+$ is linear in $t$ and $f_-$ is constant leads to $f_0$ linear in $t$:

$$f_0(t) = f_0(0) \left[ 1 + \lambda_0(t/m_\pi^2) \right].$$

(8)

With the assumption that $f_0(0) = f_+(0)$, the two parametrizations, $(\lambda_+, \xi(0))$ and $(\lambda_+, \lambda_0)$ are equivalent as long as correlation information is retained. $(\lambda_+, \lambda_0)$ correlations tend to be less strong than $(\lambda_+, \xi(0))$ correlations.

The experimental results for $\xi(0)$ and its correlation with $\lambda_+$ are listed in the $K^\pm$ and $K^0_L$ sections of the Particle Listings in section $\xi_A$, $\xi_B$, or $\xi_C$ depending on whether method A, B, or C discussed above was used. The corresponding values of $\lambda_+$ are also listed.

Because recent experiments tend to use the $(\lambda_+, \lambda_0)$ parametrization, we include a subsection for $\lambda_0$ results. Wherever possible we have converted $\xi(0)$ results into $\lambda_0$ results and vice versa.

See the 1982 version of this note [4] for additional discussion of the $K_{\mu 3}$ parameters, correlations, and conversion between parametrizations, and also for a comparison of the experimental results.

(b) $K_{e 3}$ experiments. Analysis of $K_{e 3}$ data is simpler than that of $K_{\mu 3}$ because the second term of the matrix element assuming a pure vector current [Eq. (1) above] can be neglected. Here $f_+$ is usually assumed to be linear in $t$, and the linear coefficient $\lambda_+$ of Eq. (2) is determined.
If we remove the assumption of a pure vector current, then the matrix element for the decay, in addition to the terms in Eq. (1), would contain

\[ +2m_K f_S \overline{\ell}(1 + \gamma_5)\nu \]
\[ + \frac{2f_T}{m_K} (\gamma_5) (P_\pi)_\mu \overline{\ell} \sigma_{\lambda\mu} (1 + \gamma_5)\nu , \]  

(9)

where \( f_S \) is the scalar form factor, and \( f_T \) is the tensor form factor. In the case of the \( K_{e3} \) decays where the \( f_\mu \) term can be neglected, experiments have yielded limits on \( |f_S/f_\mu| \) and \( |f_T/f_\mu| \).

References