7. ELECTROMAGNETIC RELATIONS

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<table>
<thead>
<tr>
<th>Quantity</th>
<th>Gaussian CGS</th>
<th>SI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conversion factors:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Charge:</td>
<td>2.997 921 58 × 10^{9} esu</td>
<td>= 1 C = 1 A s</td>
</tr>
<tr>
<td>Potential:</td>
<td>(1/299.729 458) statvolt</td>
<td>= 1 V = 1 J C^{-1}</td>
</tr>
<tr>
<td>Magnetic field:</td>
<td>10^4 gauss = 10^4 dyne/esu</td>
<td>= 1 T = 1 N A^{-1}m^{-1}</td>
</tr>
</tbody>
</table>

\[
\mathbf{F} = q \left( \mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right)
\]

\[
\nabla \cdot \mathbf{D} = 4\pi \rho
\]

\[
\nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} = 4\pi \mathbf{J}
\]

\[
\nabla \cdot \mathbf{B} = 0
\]

\[
\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0
\]

Constitutive relations:

\[
\mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{H} = \mathbf{B} / \mu
\]

Linear media:

\[
\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \quad \mathbf{H} = \mathbf{B} / \mu_0 - \mathbf{M}
\]

\[
\begin{align*}
\mathbf{E} &= -\nabla V - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \\
\mathbf{B} &= \nabla \times \mathbf{A}
\end{align*}
\]

\[
\begin{align*}
\mathbf{E}_0^\parallel &= \mathbf{E}_0 \\
\mathbf{E}_0^\perp &= \gamma (\mathbf{E}_0^\perp + \mathbf{v} \times \mathbf{B}) \\
\mathbf{B}_0^\parallel &= \mathbf{B}_0 \\
\mathbf{B}_0^\perp &= \gamma (\mathbf{B}_0^\perp - \frac{1}{c} \mathbf{v} \times \mathbf{E})
\end{align*}
\]

\[
\frac{1}{4\pi \varepsilon_0} = c^2 \times 10^{-7} \text{ N } \text{A}^{-2} = 8.987 \text{55...} \times 10^9 \text{ m F}^{-1} ; \quad \frac{\mu_0}{4\pi} = 10^{-7} \text{ N } \text{A}^{-1} \text{m}^{-1} ; \quad c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.997 \text{924} \times 10^8 \text{ m s}^{-1}
\]
7.1. Impedances (SI units)

\[ \rho = \text{resistivity at room temperature in } 10^{-8} \Omega \cdot \text{m} : \]
\[ \sim 1.7 \text{ for Cu} \quad \sim 5.5 \text{ for W} \]
\[ \sim 2.4 \text{ for Au} \quad \sim 73 \text{ for SS 304} \]
\[ \sim 2.8 \text{ for Al} \quad \sim 100 \text{ for Nichrome} \]
(Al alloys may have double the Al value.)

For alternating currents, instantaneous current \( I \), voltage \( V \), angular frequency \( \omega \):

\[ V = V_0 e^{j\omega t} = Z I . \quad (7.1) \]

Impedance of self-inductance \( L \): \( Z = j\omega L \).

Impedance of capacitance \( C \): \( Z = 1/j\omega C \).

Impedance of free space: \( Z = \sqrt{\mu_0/\varepsilon_0} = 376.7 \, \Omega \).

High-frequency surface impedance of a good conductor:

\[ Z = \frac{(1+j)\rho}{\delta} , \quad \text{where} \quad \delta = \text{skin depth} ; \]
\[ \delta = \sqrt{\frac{\rho}{\pi \varepsilon_0 \mu_0}} \approx \frac{6.6 \text{ cm}}{\sqrt{\nu(\text{Hz})}} \text{ for Cu} . \quad (7.3) \]

7.2. Capacitors, inductors, and transmission Lines

The capacitance between two parallel plates of area \( A \) spaced by the distance \( d \) and enclosing a medium with the dielectric constant \( \varepsilon \) is

\[ C = K \varepsilon A/d , \quad (7.4) \]

where the correction factor \( K \) depends on the extent of the fringing field. If the dielectric fills the capacitor volume without extending beyond the electrodes, the correction factor \( K \approx 0.8 \) for capacitors of typical geometry.

The inductance at high frequencies of a straight wire whose length \( l \) is much greater than the wire diameter \( d \) is

\[ L \approx 2.0 \left( \frac{\ln 1.5}{\text{cm}} \right) \cdot l \left( \ln \left( \frac{4l}{d} \right) - 1 \right) . \quad (7.5) \]

For very short wires, representative of vias in a printed circuit board, the inductance is

\[ L (\text{in nH}) = \ell/d . \quad (7.6) \]

A transmission line is a pair of conductors with inductance \( L \) and capacitance \( C \). The characteristic impedance \( Z = \sqrt{L/C} \) and the phase velocity \( v_p = 1/\sqrt{LC} = 1/\sqrt{\mu_0\varepsilon_0} \), which decreases with the inverse square root of the dielectric constant of the medium. Typical coaxial and ribbon cables have a propagation delay of about 5 ns/cm.

The impedance of a coaxial cable with outer diameter \( D \) and inner diameter \( d \) is

\[ Z = 60 \Omega \cdot \frac{1}{\sqrt{\varepsilon_r}} \ln \frac{D}{d} . \quad (7.7) \]

where the relative dielectric constant \( \varepsilon_r = \varepsilon/\varepsilon_0 \). A pair of parallel wires of diameter \( d \) and spacing \( a \) > 2.5 \( d \) has the impedance

\[ Z = 120 \Omega \cdot \frac{1}{\sqrt{\varepsilon_r}} \ln \frac{2a}{d} . \quad (7.8) \]

This yields the impedance of a wire at a spacing \( h \) above a ground plane,

\[ Z = 60 \Omega \cdot \frac{1}{\sqrt{\varepsilon_r}} \ln \frac{4h}{d} . \quad (7.9) \]

A common configuration utilizes a thin rectangular conductor above a ground plane with an intermediate dielectric (microstrip). Detailed calculations for this and other transmission line configurations are given by Gunston. *


7.3. Synchrotron radiation (CGS units)

For a particle of charge \( e \), velocity \( v = \beta c \), and energy \( E = \gamma mc^2 \), traveling in a circular orbit of radius \( R \), the classical energy loss per revolution \( \delta E \) is

\[ \delta E = \frac{4\pi e^2}{3} \beta^3 \gamma^4 . \quad (7.10) \]

For high-energy electrons or positrons (\( \beta \approx 1 \)), this becomes

\[ \delta E (\text{in MeV}) \approx 0.0885 \left[ E (\text{in GeV}) \right]^3/R (\text{in m}) . \quad (7.11) \]

For \( \gamma \gg 1 \), the energy radiated per revolution into the photon energy interval \( d[h\omega] \) is

\[ dI = \frac{8\pi}{9} \alpha \gamma F(\omega/\omega_c) d(h\omega) , \quad (7.12) \]

where \( \alpha = e^2/\hbar c \) is the fine-structure constant and

\[ \omega_c = \frac{3\gamma^3 c}{2R} . \quad (7.13) \]

is the critical frequency. The normalized function \( F(\gamma) \) is

\[ F(\gamma) = \frac{9}{8\pi} \sqrt{3} y \int_0^\infty K_{5/3}(x) \, dx , \quad (7.14) \]

where \( K_{5/3}(x) \) is a modified Bessel function of the third kind. For electrons or positrons,

\[ h\omega_c (\text{in keV}) \approx 2.22 \left[ E (\text{in GeV}) \right]^3/R (\text{in m}) . \quad (7.15) \]

Fig. 7.1 shows \( F(\gamma) \) over the important range of \( \gamma \).

Figure 7.1: The normalized synchrotron radiation spectrum \( F(\gamma) \).

For \( \gamma \gg 1 \) and \( \omega \ll \omega_c \),

\[ \frac{dI}{d(h\omega)} \approx 3.3\alpha \left( \frac{\omega R/c}{\omega_c} \right)^{1/3} , \quad (7.16) \]

whereas for

\[ \gamma \gg 1 \text{ and } \omega \gtrsim 3\omega_c , \]

\[ \frac{dI}{d(h\omega)} \approx \sqrt{\frac{3\pi}{2}} \alpha \gamma \left( \frac{\omega}{\omega_c} \right)^{1/2} e^{-\omega/\omega_c} \left[ 1 + \frac{55\omega_c}{72\omega} + \ldots \right] . \quad (7.17) \]

The radiation is confined to angles \( \lesssim 1/\gamma \) relative to the instantaneous direction of motion. For \( \gamma \gg 1 \), where Eq. (7.12) applies, the mean number of photons emitted per revolution is

\[ N_\gamma = \frac{5\pi}{\sqrt{3}} \gamma^3 , \quad (7.18) \]

and the mean energy per photon is

\[ \langle h\omega \rangle = \frac{8}{15\sqrt{3}} \gamma^2 \omega_c . \quad (7.19) \]

When \( \langle h\omega \rangle \lesssim O(E) \), quantum corrections are important.

See J.D. Jackson, Classical Electrodynamics, 3rd edition (John Wiley & Sons, New York, 1998) for more formulae and details. (Note that earlier editions had \( \omega_c \) twice as large as Eq. (7.13).)