DALITZ PLOT ANALYSIS FORMALISM

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Introduction: Weak nonleptonic decays of D and B mesons are expected to proceed dominantly through resonant two-body decays [1]; see Ref. [2] for a review of resonance phenomenology. The amplitudes are typically calculated with the Dalitz-plot analysis technique [3], which uses the minimum number of independent observable quantities. For three-body decays of a spin-0 particle to all pseudo-scalar final states, D or $B \to abc$, the decay rate [4] is

$$\Gamma = \frac{1}{(2\pi)^3 32\sqrt{s^3}} \left|\mathcal{M}\right|^2 dm_{ab}^2\, dm_{bc}^2\,, \tag{1}$$

where m_{ij} is the invariant mass of particles i and j. The coefficient of the amplitude includes all kinematic factors, and $\left|\mathcal{M}\right|^2$ contains the dynamics. The scatter plot in m_{ab}^2 versus m_{bc}^2 is the Dalitz plot. If $\left|\mathcal{M}\right|^2$ is constant, the kinematically allowed region of the plot will be populated uniformly with events. Any variation in the population over the Dalitz plot is due to dynamical rather than kinematical effects. It is straightforward to extend the formalism beyond three-body final states. For N-body final states with only spin-0 particles, phase space has dimension $3N-7$. Other decays of interest include one vector particle or a fermion/anti-fermion pair (*e.g.*, $B \to D^*\pi\pi$, $B \to \overline{\Lambda}_c p\pi$, $B \to K\ell\ell$) in the final state. For the first case, phase space has dimension $3N-5$, and for the latter two the dimension is $3N-4$.

Formalism: The amplitude for the process, $R \to rc, r \to ab$ where R is a D or B, r is an intermediate resonance, and a, b, c are pseudo-scalars, is given by

$$\mathcal{M}_r(J, L, l, m_{ab}, m_{bc}) = \sum_\lambda \langle ab|r_\lambda\rangle\, T_r(m_{ab})\, \langle cr_\lambda|R_J\rangle \tag{2}$$

$$= Z(J, L, l, \vec{p}, \vec{q}) B_L^R(|\vec{p}|) B_L^r(|\vec{q}|) T_r(m_{ab})\,.$$

The sum is over the helicity states λ of r, J is the total angular momentum of R (for D and B decays, J=0), L is the orbital angular momentum between r and c, l is the orbital angular

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momentum between a and b (the spin of r), $\vec{p}$ and $\vec{q}$ are the momenta of c and of a in the r rest frame, Z describes the angular distribution of the final-state particles, B_L^R and B_L^r are the barrier factors for the production of rc and of ab, and T_r is the dynamical function describing the resonance r. The amplitude for modeling the Dalitz plot is a phenomenological object. Differences in the parametrizations of Z, B_L, and T_r, as well as in the set of resonances r, complicate the comparison of results from different experiments.

Usually the resonances are modeled with a Breit-Wigner form, although some more recent analyses use a K-matrix formalism [5,6,7] with the P-vector approximation [8] to describe the $\pi\pi$ S-wave.

The nonresonant (NR) contribution to $D \rightarrow abc$ is parametrized as constant (S-wave) with no variation in magnitude or phase across the Dalitz plot. The available phase space is much greater for B decays, and the nonresonant contribution to $B \rightarrow abc$ requires a more sophisticated parametrization. Theoretical models of the NR amplitude [9-12] do not reproduce the distributions observed in the data. Experimentally, several parametrizations have been used [13,14].

Barrier Factor B_L: The maximum angular momentum L in a strong decay is limited by the linear momentum q. Decay particles moving slowly with an impact parameter (meson radius) d of order 1 fm have difficulty generating sufficient angular momentum to conserve the spin of the resonance. The Blatt-Weisskopf [15,16] functions B_L, given in Table 1, weight the reaction amplitudes to account for this spin-dependent effect. These functions are normalized to give $B_L = 1$ for $z = (|q|\, d)^2 = 1$. Another common formulation, B'_L, also in Table 1, is normalized to give $B'_L = 1$ for $z = z_0 = (|q_0|\, d)^2$ where q_0 is the value of q when $m_{ab} = m_r$.

Table 1: Blatt-Weisskopf barrier factors.

L	$B_L(q)$	$B'_L(q,q_0)$
0	1	1
1	$\sqrt{\dfrac{2z}{1+z}}$	$\sqrt{\dfrac{1+z_0}{1+z}}$
2	$\sqrt{\dfrac{13z^2}{(z-3)^2+9z}}$	$\sqrt{\dfrac{(z_0-3)^2+9z_0}{(z-3)^2+9z}}$
where $z=(\lvert q\rvert\, d)^2$ and $z_0=(\lvert q_0\rvert\, d)^2$		

Angular distribution: The tensor or Zemach formalism [17,18] and the helicity formalism [19,18] yield identical descriptions of the angular distributions for the decay process $R \to rc, r \to ab$ when a, b and c all have spin-0. The angular distributions for $L = 0, 1$, and 2 are given in Table 2. For final-state particles with non-zero spin (*e.g.*, radiative decays), the helicity formalism is required.

Table 2: Angular distributions for $L = 0, 1, 2$ where θ is the angle between particles a and c in the rest frame of resonance r, $\sqrt{1+\zeta^2} = E_r/m_{ab}$ is a relativistic correction, and $E_r = (m_R^2 + m_{ab}^2 - m_c^2)/2m_R$.

$J \to L + l$	Angular distribution
$0\to 0+0$	uniform
$0\to 1+1$	$(1+\zeta^2)\cos^2\theta$
$0\to 2+2$	$\left(\zeta^2+\frac{3}{2}\right)^2(\cos^2\theta-1/3)^2$

Dynamical Function T_r: The dynamical function T_r is derived from the S-matrix formalism. In general, the amplitude that a final state f couples to an initial state i is $S_{fi} = \langle f|S|i\rangle$, where the scattering operator S is unitary and satisfies $SS^\dagger = S^\dagger S = I$. The Lorentz-invariant transition operator $\hat{T}$ is defined by separating the probability that $f = i$, yielding

$$S = I + 2iT = I + 2i\left\{\rho\right\}^{1/2}\hat{T}\left\{\rho\right\}^{1/2}, \tag{3}$$

where I is the identity operator, ρ is the diagonal phase-space matrix, with $\rho_{ii} = 2q_i/m$, and q_i is the momentum of a in the r rest frame for decay channel i. In the single-channel S-wave case, $S = e^{2i\delta}$ satisfies unitarity and

$$\hat{T} = \frac{1}{\rho}e^{i\delta}\sin\delta. \tag{4}$$

There are three common formulations of the dynamical function. The Breit-Wigner formalism—the first term in a Taylor expansion about a T-matrix pole—is the simplest formulation. The K-matrix formalism [5] is more general (allowing more than one T-matrix pole and coupled channels while preserving unitarity). The Flatté distribution [20] is used to parametrize resonances near threshold and is equivalent to a one-pole, two-channel K-matrix.

Breit-Wigner Formulation: The common formulation of a Breit-Wigner resonance decaying to spin-0 particles a and b is

$$T_r(m_{ab}) = \frac{1}{m_r^2 - m_{ab}^2 - im_r\Gamma_{ab}(q)}. \tag{5}$$

The "mass-dependent" width Γ is

$$\Gamma = \Gamma_r\left(\frac{q}{q_r}\right)^{2L+1}\left(\frac{m_r}{m_{ab}}\right)B_L'(q,q_0)^2, \tag{6}$$

and $B_L'(q,q_0)$ is the Blatt-Weisskopf barrier factor from Table 1. A Breit-Wigner parametrization best describes isolated, non-overlapping resonances far from the threshold of additional decay channels. For the ρ and $\rho(1450)$ a more complex parametrization suggested by Gounaris-Sakarai [21] is often used [22-26]. Unitarity can be violated when the dynamical function is parametrized as the sum of two or more overlapping Breit-Wigners. The proximity of a threshold to a resonance distorts the line shape from a simple Breit-Wigner. Here the Flatté formula provides a better description and is discussed below.

***K*-matrix Formalism:** The T matrix can be written as

$$\hat{T} = (I - i\hat{K}\rho)^{-1}\hat{K}, \tag{7}$$

where $\hat{K}$ is the Lorentz-invariant K-matrix describing the scattering process and ρ is the phase-space factor. Resonances appear as poles in the K-matrix:

$$\hat{K}_{ij} = \sum_{\alpha} \frac{\sqrt{m_\alpha \Gamma_{\alpha i}(m) m_\alpha \Gamma_{\alpha j}(m)}}{(m_\alpha^2 - m^2)\sqrt{\rho_i \rho_j}}. \tag{8}$$

The K-matrix is real by construction, and so the associated T-matrix respects unitarity.

For a single pole in a single channel, K is

$$K = \frac{m_0\,\Gamma(m)}{m_0^2 - m^2} \tag{9}$$

and

$$T = K(1 - iK)^{-1} = \frac{m_0\Gamma(m)}{m_0^2 - m^2 - i m_0 \Gamma(m)}\,, \tag{10}$$

which is the relativistic Breit-Wigner formula. For two poles in a single channel, K is

$$K = \frac{m_\alpha \Gamma_\alpha(m)}{m_\alpha^2 - m^2} + \frac{m_\beta \Gamma_\beta(m)}{m_\beta^2 - m^2}\,. \tag{11}$$

If m_α and m_β are far apart relative to the widths, the T matrix is approximately the sum of two Breit-Wigners, $T(K_\alpha + K_\beta) \approx T(K_\alpha) + T(K_\beta)$, each of the form of Eq. (10). This approximation is not valid for two nearby resonances, in which case T can violate unitarity.

This formulation, which applies to S-channel production in two-body scattering, $ab \to cd$, can be generalized to describe the production of resonances in processes such as the decay of charm mesons. The key assumption here is that the two-body system described by the K-matrix does *not* interact with the rest of the final state [8]. The validity of this assumption varies with the production process and is appropriate for reactions such as $\pi^- p \to \pi^0\pi^0 n$ and semileptonic decays such as $D \to K\pi\ell\nu$. The assumption may be of limited validity for production processes such as $p\overline{p} \rightarrow \pi\pi\pi$ or $D \rightarrow \pi\pi\pi$. In these cases, the two-body Lorentz-invariant amplitude, $\hat{F}$, is given by

$$\hat{F}_i = (I - i\hat{K}\rho)^{-1}_{ij}\hat{P}_j = (\hat{T}\hat{K}^{-1})_{ij}\hat{P}_j\,, \tag{12}$$

where P is the production vector that parametrizes the resonance production in the open channels.

For the $\pi\pi$ S-wave, a common formulation of the K-matrix [7,24,25] is

$$K_{ij}(s) = \left[\sum_\alpha\Big(\frac{g_i^{(\alpha)}g_j^{(\alpha)}}{m_\alpha^2 - s}\Big) + f_{ij}^{sc}\frac{1-s_0^{sc}}{s-s_0^{sc}}\right]\left[\frac{(s-s_A m_\pi^2/2)(1-s_{A0})}{(s-s_{A0})}\right]. \tag{13}$$

The factor $g_i^{(\alpha)}$ is the real coupling constant of the K-matrix pole m_α to meson channel i; the parameters f_{ij}^{sc} and s_0^{sc} describe a smooth part of the K-matrix elements; the second factor in square brackets suppresses a false kinematical singularity near the $\pi\pi$ threshold (the Adler zero); and the number 1 has units GeV^2.

The production vector, with $i = 1$ denoting $\pi\pi$, is

$$P_j(s) = \left[\sum_\alpha\Big(\frac{\beta_\alpha g_j^{(\alpha)}}{m_\alpha^2 - s}\Big) + f_{1j}^{pr}\frac{1-s_0^{pr}}{s-s_0^{pr}}\right]\left[\frac{(s-s_A m_\pi^2/2)(1-s_{A0})}{(s-s_{A0})}\right]. \tag{14}$$

where the free parameters of the Dalitz plot fit are the complex production couplings β_α and the production-vector background parameters f_{1j}^{pr} and s_0^{pr}. All other parameters are fixed by scattering experiments. Ref. [6] describes the $\pi\pi$ scattering data with a 4-pole, 2-channel ($\pi\pi$, $K\bar{K}$) model, while Ref. [7] describes the scattering data with 5-pole, 5-channel ($\pi\pi$, $K\bar{K}$, $\eta\eta$, $\eta'\eta'$ and 4π) model. The former has been implemented by CLEO [27] and the latter by FOCUS [25] and BABAR [24]. In both cases, only the $\pi\pi$ channel was analyzed. A more complete coupled-channel analysis would simultaneously fit all final states accessible by rescattering.

Flatté Formalism: The Flatté formulation is used when a second channel opens close to a resonance:

$$\hat{T}(m_{ab}) = \frac{1}{m_r^2 - m_{ab}^2 - i(\rho_1 g_1^2 + \rho_2 g_2^2)}\,, \tag{15}$$

where $g_1^2 + g_2^2 = m_0\Gamma_r$. This situation occurs in the $\pi\pi$ S-wave where the $f_0(980)$ is near the $K\overline{K}$ threshold, and in the $\pi\eta$ channel where the $a_0(980)$ also lies near the $K\overline{K}$ threshold. For the $a_0(980)$ resonance, the relevant coupling constants are $g_1 = g_{\pi\eta}$ and $g_2 = g_{KK}$, and the phase space terms are $\rho_1 = \rho_{\pi\eta}$ and $\rho_2 = \rho_{KK}$, where

$$\rho_{ab} = \sqrt{\left(1 - (\frac{m_a - m_b}{m_{ab}})^2\right)\left(1 + (\frac{m_a - m_b}{m_{ab}})^2\right)}. \qquad (16)$$

For the $f_0(980)$ the relevant coupling constants are $g_1 = g_{\pi\pi}$ and $g_2 = g_{KK}$, and the phase space terms are $\rho_1 = \rho_{\pi\pi}$ and $\rho_2 = \rho_{KK}$. The charged and neutral K channels are usually assumed to have the same coupling constant but different phase space factors, due to $m_{K^+} \neq m_{K^0}$; the result is

$$\rho_{KK} = \frac{1}{2}\left(\sqrt{1-\left(\frac{2m_{K^\pm}}{m_{KK}}\right)^2} + \sqrt{1-\left(\frac{2m_{K^0}}{m_{KK}}\right)^2}\right) . \qquad (17)$$

Branching Ratios from Dalitz Plot Fits: A fit to the Dalitz plot distribution using either a Breit-Wigner or a K-matrix formalism factorizes into a resonant contribution to the amplitude $\mathcal{M}_j$ and a complex coefficient, $a_j e^{i\delta_j}$, where a_j and δ_j are real. The definition of a rate of a single process, given a set of amplitudes a_j and phases δ_j, is the square of the relevant matrix element (see Eq. (1)). The "fit fraction" is usually defined as the integral over the Dalitz plot (m_{ab} vs. m_{bc}) of a single amplitude squared divided by the integral over the Dalitz plot of the square of the coherent sum of all amplitudes, or

$$\text{fit fraction}_j = \frac{\int \left|a_j e^{i\delta_j}\mathcal{M}_j\right|^2 dm_{ab}^2 dm_{bc}^2}{\int \left|\sum_k a_k e^{i\delta_k}\mathcal{M}_k\right|^2 dm_{ab}^2 dm_{bc}^2}, \qquad (18)$$

where $\mathcal{M}_j$ is defined in Eq. (2) and described in Ref. [28]. In general, the sum of the fit fractions for all components will not be unity due to interference.

When the K-matrix of Eq. (12) is used to describe a wave (*e.g.*, the $\pi\pi$ S-wave), then $\mathcal{M}_j$ refers to the entire wave. In this case, it may not be straightforward to separate $\mathcal{M}_j$ into a sum of individual resonances unless these are narrow and well separated.

Reconstruction Efficiency and Resolution: The efficiency for reconstructing an event as a function of position on the Dalitz plot is in general non-uniform. Typically, a Monte Carlo sample generated with a uniform distribution in phase space is used to determine the efficiency. The variation in efficiency across the Dalitz plot varies with experiment and decay mode. Most recent analyses utilize a full GEANT [29] detector simulation.

Finite detector resolution can usually be safely neglected as most resonances are comparatively broad. Notable exceptions where detector resolution effects must be modeled are $\phi \to K^+K^-$, $\omega \to \pi^+\pi^-$, and $a_0 \to \eta\pi^0$. One approach is to convolve the resolution function in the Dalitz-plot variables m^2_{ab} and m^2_{bc} with the function that parametrizes the resonant amplitudes. In high-statistics data samples, resolution effects near the phase-space boundary typically contribute to a poor goodness of fit. The momenta of the final-state particles can be recalculated with a D or B mass constraint, which forces the kinematic boundaries of the Dalitz plot to be strictly respected. If the three-body mass is not constrained, then the efficiency (and the parametrization of background) may also depend on the reconstructed mass.

Backgrounds: The contribution of background to the D and B samples varies by experiment and final state. The background naturally falls into five categories: (i) purely combinatoric background containing no resonances; (ii) combinatoric background containing intermediate resonances, such as a real K^{*-} or ρ, plus additional random particles; (iii) final states containing identical particles as in $D^0 \to K^0_S\pi^0$ background to $D^0 \to \pi^+\pi^-\pi^0$ and $B \to D\pi$ background to $B \to K\pi\pi$; (iv) mistagged decays such as a real $\overline{D}^0$ or $\overline{B}^0$ incorrectly identified as a D^0 or B^0; and (v) particle misidentification of the decay products such as $D^+ \to \pi^-\pi^+\pi^+$ or $D^+_s \to K^-K^+\pi^+$ reconstructed as $D^+ \to K^-\pi^+\pi^+$.

The contribution from combinatoric background with intermediate resonances is distinct from the resonances in the signal because the former do *not* interfere with the latter since they are not from true resonances. Similarly, $D^0 \to \rho\pi$ and $D^0 \to K_S^0\pi^0$ do not interfere since strong and weak transitions proceed on different time scales. The usual identification tag of the initial particle as a D^0 or a $\overline{D}^0$ is the charge of the distinctive slow pion in the decay sequence $D^{*+} \to D^0\pi_s^+$ or $D^{*-} \to \overline{D}^0\pi_s^-$. Another possibility is the identification or "tagging" of one of the D mesons from $\psi(3770) \to D^0\overline{D}^0$, as is done for B mesons from $\Upsilon(4S)$. The mistagged background is subtle and may be mistakenly enumerated in the *signal* fraction determined by a D^0 mass fit. Mistagged decays contain true $\overline{D}^0$'s or $\overline{B}^0$'s and so the resonances in the mistagged sample exhibit interference on the Dalitz plot.

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