## 7. ELECTROMAGNETIC RELATIONS

Revised September 2005 by H.G. Spieler (LBNL).

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Gaussian CGS</th>
<th>SI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conversion factors:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Charge:</td>
<td>2.997 921 58 × 10^9 esu</td>
<td>= 1 C = 1 A s</td>
</tr>
<tr>
<td>Potential:</td>
<td>(1/298.792 458) statvolt (ergs/esu)</td>
<td>= 1 V = 1 J C^{-1}</td>
</tr>
<tr>
<td>Magnetic field:</td>
<td>10^4 gauss = 10^4 dyne/esu</td>
<td>= 1 T = 1 N A^{-1} m^{-1}</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>F = q (E + \frac{V}{c} \times B)</td>
<td>F = q (E + v \times B)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>\nabla \cdot D = 4\pi \rho</td>
<td></td>
<td>\nabla \cdot D = \rho</td>
</tr>
<tr>
<td>\nabla \times H - \frac{1}{c} \frac{\partial D}{\partial t} = 4\pi J</td>
<td>\nabla \times H - \frac{\partial D}{\partial t} = J</td>
<td></td>
</tr>
<tr>
<td>\nabla \cdot B = 0</td>
<td></td>
<td>\nabla \cdot B = 0</td>
</tr>
<tr>
<td>\nabla \times E + \frac{1}{c} \frac{\partial B}{\partial t} = 0</td>
<td>\nabla \times E + \frac{1}{c} \frac{\partial B}{\partial t} = 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Constitutive relations:</td>
<td>D = \varepsilon E + \sigma B, H = B - \mu m</td>
<td>D = \varepsilon E + \sigma B, H = B/\mu</td>
</tr>
<tr>
<td>Linear media:</td>
<td>D = \varepsilon E, H = B/\mu, 1</td>
<td>D = \varepsilon E, H = B/\mu, 1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E = - \nabla V - \frac{1}{c} \frac{\partial A}{\partial t}</td>
<td>E = - \nabla V - \frac{\partial A}{\partial t}</td>
<td></td>
</tr>
<tr>
<td>B = \nabla \times A</td>
<td>B = \nabla \times A</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>V = \sum_{\text{charges}} q_i \int \frac{\rho (r')}{</td>
<td>r - r'</td>
<td>} d^3x'</td>
</tr>
<tr>
<td>A = \frac{1}{c} \int \frac{f d\ell}{</td>
<td>r - r'</td>
<td>} - \frac{1}{c} \int \frac{J (r')}{</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E'<em>E = E</em>{</td>
<td></td>
<td>}</td>
</tr>
<tr>
<td>E'<em>E = \gamma (E</em>{\perp} + \frac{1}{c} \mathbf{v} \times \mathbf{B})</td>
<td>E'<em>E = \gamma (E</em>{\perp} + \mathbf{v} \times \mathbf{B})</td>
<td></td>
</tr>
<tr>
<td>B'<em>E = B</em>{</td>
<td></td>
<td>}</td>
</tr>
<tr>
<td>B'<em>E = \gamma (B</em>{\perp} - \frac{1}{c} \mathbf{v} \times \mathbf{E})</td>
<td>B'<em>E = \gamma (B</em>{\perp} - \mathbf{v} \times \mathbf{E})</td>
<td></td>
</tr>
</tbody>
</table>

\[
\frac{1}{4\pi \varepsilon_0} = \varepsilon^2 \times 10^{-7} \text{ N A}^{-2} = 8.987 \, 55 \ldots \times 10^8 \text{ m F}^{-1}; \quad \frac{\mu_0}{4\pi} = 10^{-7} \text{ N A}^{-1} \text{ m}^{-1}; \quad c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.997 \, 924 \, 58 \times 10^8 \text{ m s}^{-1}
\]
7.1. Impedances (SI units)

\( \rho \) = resistivity at room temperature in \( 10^{-8} \Omega \text{ m} \):
- \( \sim 1.7 \) for Cu
- \( \sim 5.5 \) for W
- \( \sim 2.4 \) for Au
- \( \sim 73 \) for SS 304
- \( \sim 2.8 \) for Al
- \( \sim 100 \) for Nichrome
(AI alloys may have double the Al value.)

For alternating currents, instantaneous current \( I \), voltage \( V \), angular frequency \( \omega \):

\[ V = V_0 e^{j\omega t} = Z I . \]  
(7.1)

Impedance of self-inductance \( L \): \( Z = j\omega L \).

Impedance of capacitance \( C \): \( Z = 1/j\omega C \).

Impedance of free space: \( Z = \sqrt{\mu_0 / \varepsilon_0} = 376.7 \Omega \).

High-frequency surface impedance of a good conductor:

\[ Z = \left( \frac{1 + j}{\delta} \right) \rho , \quad \text{where} \ \delta = \text{skin depth} ; \]  
(7.2)

\[ \delta = \sqrt{\frac{\rho}{\pi \nu \mu}} \approx \frac{6.6 \text{ cm}}{\sqrt[4]{\nu (\text{Hz})}} \text{ for Cu} . \]  
(7.3)

7.2. Capacitors, inductors, and transmission lines

The capacitance between two parallel plates of area \( A \) spaced by the distance \( d \) and enclosing a medium with the dielectric constant \( \varepsilon \) is

\[ C = K \varepsilon A/d , \]  
(7.4)

where the correction factor \( K \) depends on the extent of the fringing field. If the dielectric fills the capacitor volume without extending beyond the electrodes, the correction factor \( K \approx 0.8 \) for capacitors of typical geometry.

The inductance at high frequencies of a straight wire whose length \( l \) is much greater than the wire diameter \( d \) is

\[ L \approx 2\pi \frac{\ln h}{\text{cm}} \cdot l \left( \ln \frac{4l}{d} \right) - 1 . \]  
(7.5)

For very short wires, representative of vias in a printed circuit board, the inductance is

\[ L \text{(in nH)} \approx \ell / d . \]  
(7.6)

A transmission line is a pair of conductors with inductance \( L \) and capacitance \( C \). The characteristic impedance \( Z = \sqrt{L/C} \) and the phase velocity \( v_p = 1/\sqrt{LC} = 1/\sqrt{\mu_0 \varepsilon_0} \), which decreases with the inverse square root of the dielectric constant of the medium. Typical coaxial and ribbon cables have a propagation delay of about 5 ns/cm.

The impedance of a coaxial cable with outer diameter \( D \) and inner diameter \( d \) is

\[ Z = 60 \Omega \cdot \frac{1}{\sqrt{\varepsilon_r}} \ln \frac{D}{d} . \]  
(7.7)

where the relative dielectric constant \( \varepsilon_r = \varepsilon / \varepsilon_0 \). A pair of parallel wires of diameter \( d \) and spacing \( \alpha > 2.5 d \) has the impedance

\[ Z = 120 \Omega \cdot \frac{1}{\sqrt{\varepsilon_r}} \ln \frac{2\alpha}{d} . \]  
(7.8)

This yields the impedance of a wire at a spacing \( h \) above a ground plane,

\[ Z = 60 \Omega \cdot \frac{1}{\sqrt{\varepsilon_r}} \ln \frac{4h}{d} . \]  
(7.9)

A common configuration utilizes a thin rectangular conductor above a ground plane with an intermediate dielectric (microstrip). Detailed calculations for this and other transmission line configurations are given by Gunston.*

7.3. Synchrotron radiation (CGS units)

For a particle of charge \( e \), velocity \( v = \beta c \), and energy \( E = \gamma mc^2 \), traveling in a circular orbit of radius \( R \), the classical energy loss per revolution \( \delta E \) is

\[ \delta E = \frac{4\pi \alpha^2}{3} \frac{e^2}{R^3} \gamma^4 . \]  
(7.10)

For high-energy electrons or positrons \((\beta \approx 1)\), this becomes

\[ \delta E \text{ (in MeV)} \approx 0.0885 \frac{|E| (\text{in GeV})^3}{R (\text{in m})} . \]  
(7.11)

For \( \gamma \gg 1 \), the energy radiated per revolution into the photon energy interval \( d(\hbar \omega) \) is

\[ dI = \frac{8\pi}{9} \alpha \gamma F(\omega/\omega_c) d(\hbar \omega) , \]  
(7.12)

where \( \alpha = e^2/hc \) is the fine-structure constant and

\[ \omega_c = \frac{3\gamma^3 e^2}{2R} . \]  
(7.13)

is the critical frequency. The normalized function \( F(y) \) is

\[ F(y) = \frac{9}{8\sqrt{3}} \sqrt{y} \int_y^\infty K_{5/3}(x) \, dx , \]  
(7.14)

where \( K_{5/3}(x) \) is a modified Bessel function of the third kind. For electrons or positrons, \( \hbar \omega_c \text{ (in keV)} \approx 2.22 \frac{|E| (\text{in GeV})^3}{R (\text{in m})} . \)  
(7.15)

Fig. 7.1 shows \( F(y) \) over the important range of \( y \).

\[ \text{Figure 7.1: The normalized synchrotron radiation spectrum } F(y) . \]

For \( \gamma \gg 1 \) and \( \omega \ll \omega_c \),

\[ \frac{dI}{d(\hbar \omega)} \approx 3.3\alpha (\omega R/c)^{1/3} , \]  
(7.16)

whereas for

\[ \gamma \gg 1 \text{ and } \omega \gtrsim 3\omega_c , \]

\[ \frac{dI}{d(\hbar \omega)} \approx \frac{3\pi}{2} \alpha \gamma \left( \frac{\omega}{\omega_c} \right)^{1/2} e^{-\omega/\omega_c} \left[ 1 + \frac{55}{72} \frac{\omega_c}{\omega} + \ldots \right] . \]  
(7.17)

The radiation is confined to angles \( \lesssim 1/\gamma \) relative to the instantaneous direction of motion. For \( \gamma \gg 1 \), where Eq. (7.12) applies, the mean number of photons emitted per revolution is

\[ N_\gamma \approx \frac{5\pi}{\sqrt{3}} \gamma , \]  
(7.18)

and the mean energy per photon is

\[ \langle \hbar \omega \rangle = \frac{8}{9\sqrt{3}} \hbar \omega_c . \]  
(7.19)

When \( \langle \hbar \omega \rangle \gtrsim O(E) \), quantum corrections are important.