V_{ud}, V_{us}, THE CABIBBO ANGLE, AND CKM UNITARITY

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The Cabibbo-Kobayashi-Maskawa (CKM) [1,2] three-generation quark mixing matrix written in terms of the Wolfenstein parameters (λ, A, ρ, η) [3] nicely illustrates the orthonormality constraint of unitarity and central role played by λ.

$$V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)\ . \quad (1)$$

That cornerstone is a carryover from the two-generation Cabibbo angle, $\lambda = \sin(\theta_{\text{Cabibbo}}) = V_{us}$. Its value is a critical ingredient in determinations of the other parameters and in tests of CKM unitarity.

Unfortunately, the precise value of λ has been somewhat controversial in the past, with kaon decays suggesting [4] $\lambda \simeq 0.220$, while hyperon decays [5] and indirect determinations via nuclear β-decays imply a somewhat larger $\lambda \simeq 0.225 - 0.230$. That discrepancy is often discussed in terms of a deviation from the unitarity requirement

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1. \quad (2)$$

For many years, using a value of V_{us} derived from $K \to \pi e \nu$ (K_{e3}) decays, that sum was consistently 2–2.5 sigma below unity, a potential signal [6] for new physics effects. Below, we discuss the current status of V_{ud}, V_{us}, and their associated unitarity test in Eq. (2). (Since $|V_{ub}|^2 \simeq 1 \times 10^{-5}$ is negligibly small, it is ignored in this discussion.)

V_{ud}

The value of V_{ud} has been obtained from superallowed nuclear, neutron, and pion decays. Currently, the most precise determination of V_{ud} comes from superallowed nuclear beta-decays [6] ($0^+ \to 0^+$ transitions). Measuring their half-lives, t, and Q values which give the decay rate factor, f, leads to a precise determination of V_{ud} via the master formula [7–9]

$$|V_{ud}|^2 = \frac{2984.48(5)\,\mathrm{sec}}{ft(1+\mathrm{RC})} \tag{3}$$

where RC denotes the entire effect of electroweak radiative corrections, nuclear structure, and isospin violating nuclear effects. RC is nucleus-dependent, ranging from about +3.0% to +3.6% for the nine best measured superallowed decays. In Table 1, we give updated [10] ft values along with their implied V_{ud} for the nine best measured superallowed decays [6, 10]. They collectively give a weighted average (with errors combined in quadrature) of

$$V_{ud} = 0.97418(27)\ (\mathrm{superallowed})\ , \tag{4}$$

which, assuming unitarity, corresponds to $\lambda = 0.226(1)$. We note that the new average value of V_{ud} is shifted upward compared to our 2005 value of 0.97377(27) primarily because of a recent reevaluation of the isospin breaking Coulomb corrections by Towner and Hardy [10].

Combined measurements of the neutron lifetime, τ_n, and the ratio of axial-vector/vector couplings, $g_A \equiv G_A/G_V$, via neutron decay asymmetries can also be used to determine V_{ud}:

$$|V_{ud}|^2 = \frac{4908.7(1.9)\ \mathrm{sec}}{\tau_n(1+3g_A^2)}, \tag{5}$$

where the error stems from uncertainties in the electroweak radiative corrections [8] due to hadronic loop effects. Those effects have been recently updated and their error was reduced by about a factor of 2 [9], leading to a ± 0.0002 theoretical uncertainty in V_{ud} (common to all V_{ud} extractions). Using the world averages from this *Review*

$$\begin{aligned} \tau_n^{\mathrm{ave}} &= 885.7(8)\ \mathrm{sec} \\ g_A^{\mathrm{ave}} &= 1.2695(29) \end{aligned} \tag{6}$$

Table 1: Values of V_{ud} implied by various precisely measured superallowed nuclear beta decays. The ft values and Coulomb isospin breaking corrections are taken from Towner and Hardy [10]. Uncertainties in V_{ud} correspond to 1) nuclear structure and $Z^2\alpha^3$ uncertainties [6, 11] added in quadrature with the ft error; 2) a common error assigned to nuclear Coulomb distortion effects [11]; and 3) a common uncertainty in the radiative corrections from quantum loop effects [9]. Only the first error is used to obtain the weighted average.

Nucleus	ft (sec)	V_{ud}
^{10}C	3039.5(47)	0.97370(80)(14)(19)
^{14}O	3042.5(27)	0.97411(51)(14)(19)
^{26}Al	3037.0(11)	0.97400(24)(14)(19)
^{34}Cl	3050.0(11)	0.97417(34)(14)(19)
^{38}K	3051.1(10)	0.97413(39)(14)(19)
^{42}Sc	3046.4(14)	0.97423(44)(14)(19)
^{46}V	3049.6(16)	0.97386(49)(14)(19)
^{50}Mn	3044.4(12)	0.97487(45)(14)(19)
^{54}Co	3047.6(15)	0.97490(54)(14)(19)
Weighted Ave.		0.97418(13)(14)(19)

leads to

$$V_{ud} = 0.9746(4)_{\tau_n}(18)_{g_A}(2)_{\mathrm{RC}} \tag{7}$$

with the error dominated by g_A uncertainties (which have been expanded due to experimental inconsistencies). We note that a recent precise measurement [12] of $\tau_n = 878.5(7)(3)$ sec is also inconsistent with the world average from this *Review* and would lead to a considerably larger $V_{ud} = 0.9786(4)(18)(2)$. Future neutron studies are expected to resolve these inconsistencies and significantly reduce the uncertainties in g_A and τ_n, potentially making them the best way to determine V_{ud}.

The recently completed PIBETA experiment at PSI measured the very small ($\mathcal{O}(10^{-8})$) branching ratio for $\pi^+ \to \pi^o e^+ \nu_e$ with about $\pm 1/2\%$ precision. Their result gives [13]

$$V_{ud} = 0.9749(26) \left[\frac{BR(\pi^+ \to e^+ \nu_e(\gamma))}{1.2352 \times 10^{-4}} \right]^{\frac{1}{2}} \tag{8}$$

which is normalized using the very precisely determined theoretical prediction for $BR(\pi^+ \to e^+ \nu_e(\gamma)) = 1.2352(5) \times 10^{-4}$ [7], rather than the experimental branching ratio from this *Review* of $1.230(4) \times 10^{-4}$ which would lower the value to $V_{ud} = 0.9728(30)$. Theoretical uncertainties in that determination are very small; however, much higher statistics would be required to make this approach competitive with others.

V_{us}

$|V_{us}|$ may be determined from kaon decays, hyperon decays, and tau decays. Previous determinations have most often used $K\ell3$ decays:

$$\Gamma_{K\ell3} = \frac{G_F^2 M_K^5}{192\pi^3} S_{EW} (1 + \delta_K^\ell + \delta_{SU2}) C^2 \left| V_{us} \right|^2 f_+^2(0) I_K^\ell . \tag{9}$$

Here, ℓ refers to either e or μ, G_F is the Fermi constant, M_K is the kaon mass, S_{EW} is the short-distance radiative correction, δ_K^ℓ is the mode-dependent long-distance radiative correction, $f_+(0)$ is the calculated form factor at zero momentum transfer for the $\ell\nu$ system, and I_K^ℓ is the phase-space integral, which depends on measured semileptonic form factors. For charged kaon decays, δ_{SU2} is the deviation from one of the ratio of $f_+(0)$ for the charged to neutral kaon decay; it is zero for the neutral kaon. C^2 is 1 (1/2) for neutral (charged) kaon decays. Most determinations of $|V_{us}|$ have been based only on $K \to \pi e \nu$ decays; $K \to \pi \mu \nu$ decays have not been used because of large uncertainties in I_K^μ. The experimental measurements are the semileptonic decay widths (based on the semileptonic branching fractions and lifetime) and form factors (allowing calculation of the phase space integrals). Theory is needed for S_{EW}, δ_K^ℓ, δ_{SU2}, and $f_+(0)$.

Many new measurements during the last few years have resulted in a significant shift in V_{us}. Most importantly, recent measurements of the $K \to \pi e \nu$ branching fractions are significantly different than earlier PDG averages, probably as a result of inadequate treatment of radiation in older experiments. This effect was first observed by BNL E865 [14] in the charged kaon system and then by KTeV [15,16] in the neutral kaon system; subsequent measurements were made by KLOE [17–20], NA48 [21–23], and ISTRA+ [24]. Current averages (*e.g.*, by the PDG [25] or Flavianet [26]) of the semileptonic branching fractions are based only on recent, high-statistics experiments where the treatment of radiation is clear. In addition to measurements of branching fractions, new measurements of lifetimes [27] and form factors [28–32], have resulted in improved precision for all of the experimental inputs to V_{us}. Precise measurements of form factors for $K_{\mu 3}$ decay now make it possible to use both semileptonic decay modes to extract V_{us}.

Following the analysis of the Flavianet group [26], one finds the values of $|V_{us}|f_+(0)$ in Table 2. The average of these measurements gives

$$f_+(0)|V_{us}| = 0.21668(45). \tag{10}$$

Figure 1 shows a comparison of these results with the PDG evaluation from 2002 [33], as well as $f_+(0)(1-|V_{ud}|^2-|V_{ub}|^2)^{1/2}$, the expectation for $f_+(0)|V_{us}|$ assuming unitarity, based on $|V_{ud}| = 0.9742\pm0.0003$, $|V_{ub}| = (3.6\pm0.7)\times 10^{-3}$, and the widely used Leutwyler-Roos calculation of $f_+(0) = 0.961 \pm 0.008$ [34]. Using the result in Eq. (10) with the Leutwyler-Roos calculation of $f_+(0)$ gives

$$|V_{us}| = \lambda = 0.2255 \pm 0.0019. \tag{11}$$

Similar results for $f_+(0)$ were recently obtained from lattice gauge theory calculations [35,36]. For example, and recent 2+1 fermion dynamical wall calculation [36] gave $f_+(0) = 0.9609(51)$. Other calculations of $f_+(0)$ result in $|V_{us}|$ values that differ by as much as 2% from the result in Eq. (11). For example, a recent chiral perturbation theory calculation [37, 38] gives $f_+(0) = 0.974 \pm 0.012$, which implies a lower value of $|V_{us}| = 0.2225 \pm 0.0028$ [39].

Table 2: $|V_{us}|f_+(0)$ from $K_{\ell 3}$.

Decay Mode	$\lvert V_{us}\rvert f_+(0)$
$K^\pm e3$	0.21746 ± 0.00085
$K^\pm \mu 3$	0.21810 ± 0.00114
$K_L e3$	0.21638 ± 0.00055
$K_L \mu 3$	0.21678 ± 0.00067
$K_S e3$	0.21554 ± 0.00142
Average	0.21668 ± 0.00045

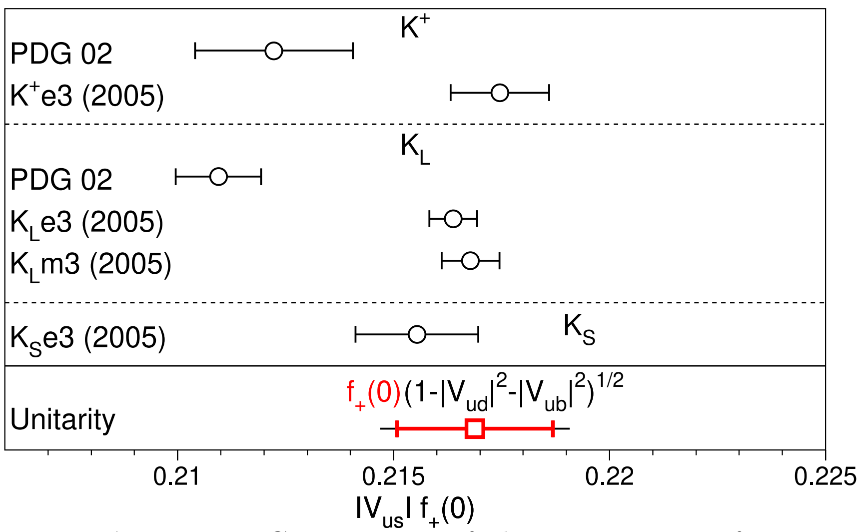

Figure 1: Comparison of determinations of $|V_{us}|f_+(0)$ from this review (labeled 2005), from the PDG 2002, and with the prediction from unitarity using $|V_{ud}|$ and the Leutwyler-Roos calculation of $f_+(0)$ [34]. For $f_+(0)(1-|V_{ud}|^2-|V_{ub}|^2)^{1/2}$, the inner error bars are from the quoted uncertainty in $f_+(0)$; the total uncertainties include the $|V_{ud}|$ and $|V_{ub}|$ errors. See full-color version on color pages at end of book.

A value of V_{us} can also be obtained from a comparison of the radiative inclusive decay rates for $K \to \mu\nu(\gamma)$ and $\pi \to \mu\nu(\gamma)$ combined with a lattice gauge theory calculation of f_K/f_π via [40]

$$\frac{|V_{us}|f_K}{|V_{ud}|f_\pi} = 0.2387(4)\left[\frac{\Gamma(K\to\mu\nu(\gamma))}{\Gamma(\pi\to\mu\nu(\gamma))}\right]^{\frac{1}{2}} \tag{12}$$

with the small error coming from electroweak radiative corrections. Employing

$$\frac{\Gamma(K\to\mu\nu(\gamma))}{\Gamma(\pi\to\mu\nu(\gamma))} = 1.3337(46), \tag{13}$$

which averages in the KLOE result [41], $B(K\to\mu\nu(\gamma)) = 63.66(9)(15)\%$ and [42, 43]

$$f_K/f_\pi = 1.208(2)(+7/-14) \tag{14}$$

along with the value of V_{ud} in Eq. (4) leads to

$$|V_{us}| = 0.2223(5)(1.208 f_\pi/f_K). \tag{15}$$

It should be mentioned that hyperon decay fits suggest [5]

$$|V_{us}| = 0.2250(27)\ \text{Hyperon Decays} \tag{16}$$

modulo SU(3) breaking effects that could shift that value up or down. We note that a recent representative effort [44] that incorporates SU(3) breaking found $V_{us} = 0.226(5)$. Similarly, strangeness changing tau decays give [45]

$$|V_{us}| = 0.2208(34)\ \text{Tau Decays} \tag{17}$$

where the central value depends on the strange quark mass.

Employing the value of V_{ud} in Eq. (4) and V_{us} in Eq. (11) leads to the unitarity consistency check

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9999(5)(9), \tag{18}$$

where the first error is the uncertainty from $|V_{ud}|^2$ and the second error is the uncertainty from $|V_{us}|^2$. The result is in good agreement with unitarity. Averaging the direct determination of λ (V_{us}) with the determination derived from unitarity and V_{ud} gives $\lambda = 0.226(1)$. Although unitarity now seems well established, issues regarding the Q values in superallowed nuclear β-decays, $\tau_n, g_A, f_+(0)$ and f_K/f_π must still be resolved before a definitive confirmation is possible.

CKM Unitarity Constraints

The current good experimental agreement with unitarity, $|V_{ud}|^2+|V_{us}|^2+|V_{ub}|^2 = 0.9999(10)$ provides strong confirmation of Standard Model radiative corrections (which range between 3-4% depending on the nucleus used) at better than the 30 sigma level [46]. In addition, it implies constraints on "New Physics" effects at both the tree and quantum loop levels. Those effects could be in the form of contributions to nuclear beta decays, $K_{\ell 3}$ decays and/or muon decays, with the last of these providing normalization via the muon lifetime [47], which is used to obtain the Fermi constant, $G_\mu = 1.166371(6) \times 10^{-5}\mathrm{GeV}^{-2}$.

We illustrate the implications of CKM unitarity for: 1) exotic muon decays [48](beyond ordinary muon decay $\mu^+ \to e^+\nu_e\bar{\nu}_\mu$); and 2) new heavy quark mixing V_{uD} [49]. Other examples in the literature [50,51] include Z_χ boson quantum loop effects, supersymmetry, leptoquarks, compositeness etc.

Exotic Muon Decays

If additional lepton flavor violating decays such as $\mu^+ \to e^+\bar{\nu}_e\nu_\mu$ (wrong neutrinos) occur, they would cause confusion in searches for neutrino oscillations at, for example, muon storage rings/neutrino factories or other neutrino sources from muon decays. Calling the rate for all such decays $\Gamma(\text{exotic } \mu \text{ decays})$, they should be subtracted before the extraction of G_μ and normalization of the CKM matrix. Since that is not done and unitarity works, one has (at one-sided 95% CL)

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 - BR(\text{exotic } \mu \text{ decays}) \geq 0.9982 \tag{19}$$

or

$$BR(\text{exotic } \mu \text{ decays}) < 0.0018 \; . \tag{20}$$

That bound is a factor of 6–7 better than the direct experimental bound on $\mu^+ \to e^+\bar{\nu}_e\nu_\mu$.

New Heavy Quark Mixing

Heavy D quarks naturally occur in fourth quark generation models and some heavy quark "new physics" scenarios such as E_6 grand unification. Their mixing with ordinary quarks gives rise to V_{ud} which is constrained by unitarity (one sided 95% CL)

$$\begin{aligned}|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 - |V_{uD}|^2 > 0.9982 \\ |V_{uD}| < 0.04 \ . \end{aligned} \tag{21}$$

A similar constraint applies to heavy neutrino mixing and the couplings $V_{\mu N}$ and V_{eN}.

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