### 7. ELECTROMAGNETIC RELATIONS

Revised September 2005 by H.G. Spieler (BNL).

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Gaussian CGS</th>
<th>SI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Charge:</td>
<td>2.997 921 58 × 10⁹ esu</td>
<td>= 1 C = 1 A s</td>
</tr>
<tr>
<td>Potential:</td>
<td>(1/299.792 458) statvolt (ergs/esu)</td>
<td>= 1 V = 1 J C⁻¹</td>
</tr>
<tr>
<td>Magnetic field:</td>
<td>10⁴ gauss = 10⁴ dyne/esu</td>
<td>= 1 T = 1 N A⁻¹ m⁻¹</td>
</tr>
</tbody>
</table>

\[
\mathbf{F} = q (\mathbf{E} + \frac{\nabla}{c} \times \mathbf{B})
\]

\[
\mathbf{D} = \varepsilon \varepsilon_0 \mathbf{E}
\]

\[
\mathbf{B} = \frac{\mu_0}{c} \mathbf{H}
\]

### Constitutive relations:

\[\mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{H} = \mathbf{B}/\mu\]

\[\mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{H} = \mathbf{B}/\mu\]

\[\epsilon_0 = 8.854 187 \ldots \times 10^{-12} \text{ F m}^{-1}\]

\[\mu_0 = 4\pi \times 10^{-7} \text{ N A}^{-2}\]

\[\mathbf{E} = -\nabla V - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}\]

\[\mathbf{B} = \nabla \times \mathbf{A}\]

\[\mathbf{E}'_\parallel = \mathbf{E}_\parallel\]

\[\mathbf{E}'_\perp = \gamma (\mathbf{E}_\perp + \frac{1}{c} \mathbf{v} \times \mathbf{B})\]

\[\mathbf{B}'_\parallel = \mathbf{B}_\parallel\]

\[\mathbf{B}'_\perp = \gamma (\mathbf{B}_\perp - \frac{1}{c} \mathbf{v} \times \mathbf{E})\]

\[\frac{1}{4\pi \varepsilon_0} = c^2 \times 10^{-7} \text{ N A}^{-2} = 8.987 55 \ldots \times 10^9 \text{ m F}^{-1}; \quad \frac{\mu_0}{4\pi} = 10^{-7} \text{ N A}^{-2}; \quad c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.997 924 \ 58 \times 10^8 \text{ m s}^{-1}\]
7.1. Impedances (SI units)

\( \rho = \text{resistivity at room temperature in } 10^{-8} \, \Omega \, \text{m} \):
\( \sim 1.7 \) for Cu \( \sim 5.5 \) for W
\( \sim 2.4 \) for Au \( \sim 73 \) for SS 304
\( \sim 2.8 \) for Al \( \sim 100 \) for Nichrome

For alternating currents, instantaneous current \( I \), voltage \( V \), angular frequency \( \omega \):
\[
V = V_0 \, e^{j\omega t} = ZI .
\tag{7.1}
\]

Impedance of self-inductance \( L \): \( Z = j\omega L \).

Impedance of capacitance \( C \): \( Z = 1/j\omega C \).

Impedance of free space: \( Z = \sqrt{\mu_0/\varepsilon_0} = 376.7 \, \Omega \).

High-frequency surface impedance of a good conductor:
\[
Z = \left(1 + j\frac{\rho}{\varepsilon_0} \right) \frac{d}{\delta} ,
\tag{7.2}
\]
\[
\delta = \sqrt{\frac{\rho}{\varepsilon_0 \mu_0}} \approx 6.6 \, \text{cm} \quad \text{for Cu}.
\tag{7.3}
\]

7.2. Capacitors, inductors, and transmission Lines

The capacitance between two parallel plates of area \( A \) spaced by the distance \( d \) and enclosing a medium with the dielectric constant \( \varepsilon \) is
\[
C = K \varepsilon A/d ,
\tag{7.4}
\]
where the correction factor \( K \) depends on the extent of the fringing field. If the dielectric fills the capacitor volume without extending beyond the electrodes, the correction factor \( K \approx 0.8 \) for capacitors of typical geometry.

The inductance at high frequencies of a straight wire whose length \( \ell \) is much greater than the wire diameter \( d \) is
\[
L \approx 2.0 \frac{\text{mH}}{\text{cm}} \cdot \ell \left( \ln \frac{4\ell}{d} - 1 \right).
\tag{7.5}
\]

For very short wires, representative of vias in a printed circuit board, the inductance is
\[
L \approx \ell/d .
\tag{7.6}
\]

A transmission line is a pair of conductors with inductance \( L \) and capacitance \( C \). The characteristic impedance \( Z = \sqrt{L/C} \) is independent of the phase velocity \( v_p = 1/\sqrt{LC} = 1/\sqrt{\mu_0 \varepsilon_0} \), which decreases with the inverse square root of the dielectric constant of the medium. Typical coaxial and ribbon cables have a propagation delay of about 5 ns/cm. The impedance of a coaxial cable with outer diameter \( D \) and inner diameter \( d \) is
\[
Z = 60 \Omega \cdot \frac{1}{\sqrt{\varepsilon_r}} \ln \frac{D}{d}.
\tag{7.7}
\]

where the relative dielectric constant \( \varepsilon_r = \varepsilon/\varepsilon_0 \). A pair of parallel wires of diameter \( d \) and spacing \( a > 2.5 \, d \) has the impedance
\[
Z = 120 \Omega \cdot \frac{1}{\sqrt{\varepsilon_r}} \ln \frac{2a}{d}.
\tag{7.8}
\]

This yields the impedance of a wire at a spacing \( h \) above a ground plane,
\[
Z = 60 \Omega \cdot \frac{1}{\sqrt{\varepsilon_r}} \ln \frac{4h}{d}.
\tag{7.9}
\]

A common configuration utilizes a thin rectangular conductor above a ground plane with an intermediate dielectric (microstrip). Detailed calculations for this and other transmission line configurations are given by Gunston.*

7.3. Synchrotron radiation (CGS units)

For a particle of charge \( e \), velocity \( v = \beta c \), and energy \( E = \gamma mc^2 \), traveling in a circular orbit of radius \( R \), the classical energy loss per revolution \( \delta E \) is
\[
\delta E = \frac{4\pi e^2}{3} \frac{1}{R} \frac{e^3 \beta^4}{c^6}.
\tag{7.10}
\]

For high-energy electrons or positrons \( (\beta \approx 1) \), this becomes
\[
\delta E \approx \frac{0.0885}{E} \frac{\text{MeV}}{R} \frac{\text{MeV}}{R}.
\tag{7.11}
\]

For \( \gamma \gg 1 \), the energy radiated per revolution into the photon energy interval \( d(\hbar \omega) \) is
\[
dI = \frac{8\pi}{9} \alpha \gamma F(\omega/\omega_c) d(\hbar \omega) ,
\tag{7.12}
\]
where \( \alpha = e^2/\hbar c \) is the fine-structure constant and
\[
\omega_c = \frac{3\gamma^3 e^2}{2 R}.
\tag{7.13}
\]

is the critical frequency. The normalized function \( F(\omega) \) is
\[
F(\omega) = \frac{9}{8\pi} \sqrt{3} \int_{\omega_c}^{\infty} K_3(\xi) \, d\xi ,
\tag{7.14}
\]
where \( K_3(\xi) \) is a modified Bessel function of the third kind. For electrons or positrons,
\[
\hbar \omega_c \approx 2.22 \frac{\text{MeV}}{\text{m}} \cdot \frac{E}{\text{MeV}}.
\tag{7.15}
\]

Fig. 7.1 shows \( F(\omega) \) over the important range of \( \omega_c \).

![Diagram of synchrotron radiation spectrum](image)

\( F(\omega) \) has the units of \( 1/(\text{MeV} \cdot \text{m}) \), and the energy per photon is
\[
(\hbar \omega) = \frac{8 \pi}{15 \gamma^3} \hbar \omega_c.
\tag{7.19}
\]

When \( (\hbar \omega) \gtrsim O(E) \), quantum corrections are important.

See J.D. Jackson, *Classical Electrodynamics*, 3rd edition (John Wiley & Sons, New York, 1998) for more formulas and details. (Note that earlier editions had \( \omega_c \) twice as large as Eq. (7.13).)

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