

40. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over every coefficient, e.g., for $-8/15$ read $-\sqrt{8/15}$.

Notation:

J	J	...
M	M	...
m_1	m_2	
m_1	m_2	Coefficients
\vdots	\vdots	
\vdots	\vdots	

$1/2 \times 1/2$

1		
+1/2	-1/2	1
-1/2	+1/2	1
-1/2	-1/2	1

$1 \times 1/2$

3/2	3/2	1/2
+1	+1/2	1
+1	-1/2	1/3
0	+1/2	2/3
0	-1/2	2/3
-1	+1/2	1/3
-1	-1/2	1

2×1

3	3	2
+2	+1	1
+2	0	1/3
+1	+1	2/3
+1	0	2/3
0	+1	1/3
0	0	2/3
-1	+1	1/3
-1	0	2/3
-2	+1	1/3
-2	0	2/3

1×1

2	2	1
+1	+1	1
+1	0	1/2
0	+1	1/2
0	0	1
-1	+1	1/2
-1	0	1/2
-2	+1	1/2
-2	0	1

$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos \theta$

$Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}$

$Y_2^0 = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$

$Y_2^1 = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi}$

$Y_2^2 = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{2i\phi}$

$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$

$2 \times 1/2$

5/2	5/2	3/2
+2	+1/2	1
+2	-1/2	1/5
+1	+1/2	4/5
+1	-1/2	4/5
0	+1/2	2/5
0	-1/2	3/5
-1	+1/2	3/5
-1	-1/2	2/5
-2	+1/2	1/5
-2	-1/2	1/5

$3/2 \times 1/2$

2	2	1
+3/2	+1/2	1
+3/2	-1/2	1/4
+1/2	+1/2	3/4
+1/2	-1/2	3/4
0	+1/2	1/4
0	-1/2	3/4
-1	+1/2	1/4
-1	-1/2	3/4
-2	+1/2	1/4
-2	-1/2	3/4

$3/2 \times 1$

5/2	5/2	3/2
+3/2	+1	1
+3/2	0	2/5
+1/2	+1	3/5
+1/2	0	3/5
0	+1	1/5
0	0	2/5
-1	+1	1/5
-1	0	2/5
-2	+1	1/5
-2	0	2/5

$3/2 \times 3/2$

3	3	2
+3/2	+3/2	1
+3/2	+1/2	1/2
+1/2	+3/2	1/2
+1/2	-1/2	3/10
-1/2	+3/2	3/10
-1/2	-1/2	3/10
0	+3/2	1/5
0	-1/2	2/5
0	0	1/5
-1	+3/2	1/5
-1	-1/2	2/5
-2	+3/2	1/5
-2	-1/2	2/5

$d_{\ell m,0}^\ell = \sqrt{\frac{4\pi}{2\ell+1}} Y_\ell^m e^{-im\phi}$

$(j_1 j_2 m_1 m_2 | j_1 j_2 J M)$
 $= (-1)^{J-j_1-j_2} (j_2 j_1 m_2 m_1 | j_2 j_1 J M)$

$d_{m',m}^j = (-1)^{m-m'} d_{m,m'}^j = d_{-m,-m'}^j$

$2 \times 3/2$

7/2	7/2	5/2
+2	+3/2	1
+2	+1/2	3/7
+1	+3/2	4/7
+1	-1/2	4/7
0	+3/2	1/7
0	-1/2	16/35
-1	+3/2	4/7
-1	-1/2	16/35
-2	+3/2	1/7
-2	-1/2	16/35

2×2

4	4	3
+2	+2	1
+2	+1	1/2
+1	+2	1/2
+1	0	3/4
0	+2	4/7
0	0	3/4
-1	+2	1/2
-1	0	3/4
-2	+2	1/2
-2	0	3/4

$d_{1,0}^1 = \cos \theta$

$d_{1/2,1/2}^{1/2} = \cos \frac{\theta}{2}$

$d_{1/2,-1/2}^{1/2} = -\sin \frac{\theta}{2}$

$d_{1,1}^1 = \frac{1 + \cos \theta}{2}$

$d_{1,0}^1 = -\frac{\sin \theta}{\sqrt{2}}$

$d_{1,-1}^1 = \frac{1 - \cos \theta}{2}$

$3/2 \times 3/2$

3	3	2
+3/2	+3/2	1
+3/2	+1/2	1/2
+1/2	+3/2	1/2
+1/2	-1/2	3/10
-1/2	+3/2	3/10
-1/2	-1/2	3/10
0	+3/2	1/5
0	-1/2	2/5
0	0	1/5
-1	+3/2	1/5
-1	-1/2	2/5
-2	+3/2	1/5
-2	-1/2	2/5

2×2

4	4	3
+2	+2	1
+2	+1	1/2
+1	+2	1/2
+1	0	3/4
0	+2	4/7
0	0	3/4
-1	+2	1/2
-1	0	3/4
-2	+2	1/2
-2	0	3/4

$d_{3/2,3/2}^{3/2} = \frac{1 + \cos \theta}{2} \cos \frac{\theta}{2}$

$d_{3/2,1/2}^{3/2} = -\sqrt{3} \frac{1 + \cos \theta}{2} \sin \frac{\theta}{2}$

$d_{3/2,-1/2}^{3/2} = \sqrt{3} \frac{1 - \cos \theta}{2} \cos \frac{\theta}{2}$

$d_{3/2,-3/2}^{3/2} = -\frac{1 - \cos \theta}{2} \sin \frac{\theta}{2}$

$d_{1/2,1/2}^{3/2} = \frac{3 \cos \theta - 1}{2} \cos \frac{\theta}{2}$

$d_{1/2,-1/2}^{3/2} = -\frac{3 \cos \theta + 1}{2} \sin \frac{\theta}{2}$

$d_{2,2}^2 = \left(\frac{1 + \cos \theta}{2} \right)^2$

$d_{2,1}^2 = -\frac{1 + \cos \theta}{2} \sin \theta$

$d_{2,0}^2 = \frac{\sqrt{6}}{4} \sin^2 \theta$

$d_{2,-1}^2 = -\frac{1 - \cos \theta}{2} \sin \theta$

$d_{2,-2}^2 = \left(\frac{1 - \cos \theta}{2} \right)^2$

$d_{1,1}^2 = \frac{1 + \cos \theta}{2} (2 \cos \theta - 1)$

$d_{1,0}^2 = -\sqrt{\frac{3}{2}} \sin \theta \cos \theta$

$d_{1,-1}^2 = \frac{1 - \cos \theta}{2} (2 \cos \theta + 1)$

$d_{0,0}^2 = \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$

Figure 40.1: The sign convention is that of Wigner (Group Theory, Academic Press, New York, 1959), also used by Condon and Shortley (The Theory of Atomic Spectra, Cambridge Univ. Press, New York, 1953), Rose (Elementary Theory of Angular Momentum, Wiley, New York, 1957), and Cohen (Tables of the Clebsch-Gordan Coefficients, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).