17. HEAVY-QUARK AND SOFT-COLLINEAR EFFECTIVE THEORY

Updated September 2015 by C.W. Bauer (LBNL) and M. Neubert (U. Mainz).

17.1. Effective Field Theories

Quantum field theories provide the most precise computational tools for describing physics at the highest energies. One of their characteristic features is that they almost inevitably involve multiple length scales. When trying to determine the value of an observable, quantum field theory demands that all possible virtual states and hence all particles be included in the calculation. Since these particles have widely different masses, the final prediction is sensitive to many scales. This fact represents a formidable challenge from a practical point of view. No realistic quantum field theories can be solved exactly, so that one needs to resort to approximation schemes; these, however, are typically most straightforward when only a single scale is involved at a time.

Effective field theories (EFTs) provide a general theoretical framework to deal with the multi-scale problems of realistic quantum field theories. This framework aims at reducing such problems to a combination of separate and simpler single-scale problems; simultaneously, however, it provides an organization scheme whereby the other scales are not omitted but allowed to play their role in a separate step of the computation. The philosophy and basic principles of this approach are very generic, and correspondingly EFTs represent a widely used method in many different areas of high-energy physics, from the low-energy scales of atomic and nuclear physics to the high-energy scales of (partly yet unknown) elementary-particle physics, see [1–3] for some early references. EFTs can play a role both within analytic perturbative computations and in the context of non-perturbative numerical simulations; One of the simplest applications of EFTs to particle physics concerns the description of an underlying theory that is only probed at energy scales $E < \Lambda$. Any particle with mass $m > \Lambda$ cannot be produced as a real state and therefore only leads to short-distance virtual effects. Thus, one can construct an effective theory in which the quantum fluctuations of such heavy particles are "integrated out" from the generating functional for Green functions. This results in a simpler theory containing only those degrees of freedom that are relevant to the energy scales under consideration. In fact, the standard model of particle physics itself is widely viewed as an EFT of some yet unknown, more fundamental theory.

The development of any effective theory starts by identifying the degrees of freedom that are relevant to describe the physics at a given energy (or length) scale and constructing the Lagrangian describing the interactions among these fields. Short-distance quantum fluctuations associated with much smaller length scales are absorbed into the coefficients of the various operators in the effective theory. These coefficients are determined in a matching procedure, by requiring that the EFT reproduces the matrix elements of the full theory up to power corrections. In many cases the effective Lagrangian exhibits enhanced symmetries compared with the fundamental theory, allowing for simple and sometimes striking predictions relating different observables.

17.2. Heavy-Quark Effective Theory

Heavy-quark systems provide prime examples for applications of the EFT technology, because the hierarchy $m_Q \gg \Lambda_{\text{QCD}}$ (with $Q = b, c$) provides a natural separation of scales. Physics at the scale m_Q is of a short-distance nature and can be treated perturbatively, while for heavy-quark systems there is always also some hadronic physics governed by the confinement scale Λ_{QCD} of the strong interaction. Being able to separate the short-distance and long-distance effects associated with these two scales is crucial for any quantitative description. For instance, if the long-distance hadronic matrix elements are obtained from lattice QCD, then it is necessary to analytically compute the effects of short-wavelength modes that do not fit on the lattice. In many other instances, the long-distance physics can be encoded in a small number of hadronic parameters.

17.2.1. ***General idea and derivation of the effective Lagrangian*** : The simplest effective theory for heavy-quark systems is the heavy-quark effective theory (HQET) [4–7] (see [8,9] for detailed discussions). It provides a simplified description of the soft interactions of a single heavy quark with light partons. This includes the interactions that bind the heavy quark with other light partons inside heavy mesons and baryons.

A softly interacting heavy quark is nearly on-shell. Its momentum may be decomposed as $p_Q = m_Q v + k$, where v is the 4-velocity of the hadron containing the heavy quark. The "residual momentum" k results from the soft interactions of the heavy quark with its environment and satisfies $v \cdot k \sim \Lambda_{\text{QCD}}$ and $k^2 \sim \Lambda_{\text{QCD}}^2$, which in the rest frame of the heavy hadron reduces to $k^\mu \sim \Lambda_{\text{QCD}}$. In the limit $m_Q \gg \Lambda_{\text{QCD}}$, the soft interactions do not change the 4-velocity of the heavy quark, which is therefore a conserved quantum number that is often used as a label on the effective heavy-quark fields. A nearly on-shell Dirac spinor has two large and two small components. We define

$$Q(x) = e^{-im_Q v\cdot x} \left[h_v(x) + H_v(x) \right], \tag{17.1}$$

where

$$h_v(x) = e^{im_Q v\cdot x}\, \frac{1+\not{v}}{2}\, Q(x)\,, \qquad H_v(x) = e^{im_Q v\cdot x}\, \frac{1-\not{v}}{2}\, Q(x) \tag{17.2}$$

are the large ("upper") and small ("lower") components of the spinor field, respectively. The extraction of the phase factor in Eq. (17.1) implies that the fields h_v and H_v carry the residual momentum k. The field H_v is $1/m_Q$ suppressed relative to h_v and describes quantum fluctuations far off the mass shell. Integrating it out using its equations of motion yields the HQET Lagrangian

$$\mathcal{L}_{\text{HQET}} = \bar{h}_v\, iv\cdot D_s\, h_v + \frac{1}{2m_Q} \left[\bar{h}_v (iD_s)^2 h_v + C_{\text{mag}}(\mu)\, \frac{g}{2}\, \bar{h}_v\, \sigma_{\mu\nu}\, G_s^{\mu\nu} h_v \right] + \ldots\,. \tag{17.3}$$

The covariant derivative $iD_s^\mu = i\partial^\mu + gA_s^\mu$ and the field strength $G_s^{\mu\nu}$ contain only the soft gluon field. Hard gluons have been integrated out, and their effects are contained in the Wilson coefficients of the operators in the effective Lagrangian. From the leading operator one derives the Feynman rules of HQET. The new operators entering at subleading order are referred to as the "kinetic energy" and "chromo-magnetic interaction". The kinetic-energy operator corresponds to the first correction term in the Taylor expansion of the relativistic energy $E = m_Q + \vec{p}^{\,2}/2m_Q + \ldots$. Lorentz invariance, which is encoded as a reparametrization invariance of the effective Lagrangian [10], ensures that its Wilson coefficient is not renormalized ($C_{\rm kin} \equiv 1$). The coefficient $C_{\rm mag}$ of the chromo-magnetic operator receives corrections starting at one-loop order.

17.2.2. ***Spin-flavor symmetry*** : The leading term in the HQET Lagrangian exhibits a global spin-flavor symmetry. Its physical meaning is that, in the infinite mass limit, the properties of hadronic systems containing a single heavy quark are insensitive to the spin and flavor of the heavy quark [11,12]. The spin symmetry results from the fact that there are no Dirac matrices in the leading term of the effective Lagrangian in Eq. (17.3), implying that the interactions of the heavy quark with soft gluons leave its spin unchanged. The flavor symmetry arises since the mass of the heavy quark does not appear at leading order. For n_Q heavy quarks moving at the same velocity, one can simply extend Eq. (17.3) by summing over n_Q identical terms for heavy-quark fields h_v^i. The result is invariant under rotations in flavor space. When combined with the spin symmetry, the symmetry group becomes promoted to SU($2n_Q$). These symmetries are broken by the operators at subleading power in the $1/m_Q$ expansion.

The spin-flavor symmetry leads to many interesting relations between the properties of hadrons containing a heavy quark. The most direct consequences concern the spectroscopy of such states [13]. In the heavy-quark limit, the spin of the heavy quark and the total angular momentum j of the light degrees of freedom are separately conserved by the strong interactions. Because of heavy-quark symmetry, the dynamics is independent of the spin and mass of the heavy quark. Hadronic states can thus be classified by the quantum numbers (flavor, spin, parity, etc.) of the light degrees of freedom. The spin symmetry predicts that, for fixed $j \neq 0$, there is a doublet of degenerate states with total spin $J = j \pm 1/2$. The flavor symmetry relates the properties of states with different heavy-quark flavor.

17.2.3. ***Weak decay form factors*** : Of particular interest are the relations between the weak decay form factors of heavy mesons, which parametrize hadronic matrix elements of currents between two mesons containing a heavy quark. These relations have been derived by Isgur and Wise [12], generalizing ideas developed by Nussinov and Wetzel [14] and Voloshin and Shifman [15]. For the purpose of this discussion, it is convenient to work with a mass-independent normalization of meson states and use velocity rather than momentum variables.

Consider the elastic scattering of a pseudoscalar meson, $P(v) \to P(v')$, induced by an external vector current coupled to the heavy quark contained in P, which acts as a color source moving with the meson's velocity v. The action of the current is to replace instantaneously the color source by one moving at velocity v'. Soft gluons need to be exchanged in order to rearrange the light degrees of freedom and build up the final state meson moving at velocity v'. This rearrangement leads to a form-factor suppression. The important observation is that, in the $m_Q \to \infty$ limit, the form factor can only depend on the Lorentz boost $\gamma = v \cdot v'$ connecting the rest frames of the initial and final-state mesons (as long as $\gamma = \mathcal{O}(1)$). In the effective theory the hadronic matrix element describing the scattering process can therefore be written as

$$\langle P(v')|\,\bar{h}_{v'}\gamma^\mu h_v\,|P(v)\rangle = \xi(v\cdot v')(v+v')^\mu, \tag{17.4}$$

with a form factor $\xi(v\cdot v')$ that is real and independent of m_Q. By flavor symmetry, the form factor remains identical when one replaces the heavy quark Q in one of the meson states by a heavy quark Q' of a different flavor, thereby turning P into another pseudoscalar meson P'. At the same time, the current becomes a flavor-changing vector current. This universal form factor is called the Isgur-Wise function [12]. For equal velocities the vector current $J^\mu = \bar{h}_v\gamma^\mu h_v$ is conserved in the effective theory, irrespective of the flavor of the heavy quarks. The corresponding conserved charges are the generators of the flavor symmetry. It follows that the Isgur-Wise function is normalized at the point of equal velocities: $\xi(1)=1$. Since the recoil energy of the daughter meson P' in the rest frame of the parent meson P is $E_{\rm recoil} = m_{P'}\,(v\cdot v'-1)$, the point $v\cdot v'=1$ is referred to as the zero-recoil limit. The heavy-quark spin symmetry leads to additional relations among weak decay form factors. It can be used to relate matrix elements involving vector mesons to those involving pseudoscalar mesons, which once again can be described completely in terms of the universal Isgur-Wise function.

The form factor relations imposed by heavy-quark symmetry describe the semileptonic decay processes $\bar{B}\to D\,\ell\,\bar\nu$ and $\bar{B}\to D^*\ell\,\bar\nu$ in the limit of infinite heavy-quark masses. They are model-independent consequences of QCD. The known normalization of the Isgur-Wise function at zero recoil can be used to obtain a model-independent measurement of the element $|V_{cb}|$ of the Cabibbo-Kobayashi-Maskawa (CKM) matrix. The semileptonic decay $\bar{B}\to D^*\ell\,\bar\nu$ is particularly well suited for this purpose [16]. Experimentally this is a very clean mode, since the reconstruction of the D^* meson mass provides a powerful rejection against background. From the theoretical point of view, it is ideal since the decay rate at zero recoil is protected by Luke's theorem against first-order power corrections in $1/m_Q$ [17]. This is described in more detail in Section 12. Corrections to the heavy-quark symmetry relations for the $\bar{B}\to D^{(*)}$ form factors near zero recoil can also be constrained using sum rules derived in the small-velocity limit [18,19].

17.2.4. ***Decoupling transformation*** : At leading order in $1/m_Q$, the couplings of soft gluons to heavy quarks in the effective Lagrangian Eq. (17.3) can be removed by the field redefinition $h_v(x) = Y_v(x)\,h_v^{(0)}(x)$, where $Y_v(x)$ is a soft Wilson line along the direction of v, extending from minus infinity to the point x. In terms of the new fields the leading-order HQET Lagrangian becomes $\mathcal{L}_{\rm HQET} = \bar{h}_v^{(0)} iv\cdot\partial\, h_v^{(0)}$. It describes a free theory as far as the strong interactions of heavy quarks are concerned. However, the theory is nevertheless non-trivial in the presence of external sources. Consider, e.g., the case of a weak-interaction heavy-quark current

$$\bar{h}_{v'}\gamma^\mu(1-\gamma_5)h_v = \bar{h}_{v'}^{(0)}\gamma^\mu(1-\gamma_5)\,Y_{v'}^\dagger Y_v\,h_v^{(0)}, \tag{17.5}$$

where v and v' are the velocities of the heavy mesons containing the heavy quarks. Unless the two velocities are equal, corresponding to the zero-recoil limit discussed above, the object $Y_{v'}^\dagger Y_v$ is non-trivial, and hence the soft gluons do not decouple from the heavy quarks inside the current operator. One may interpret $Y_{v'}^\dagger Y_v$ as a Wilson loop with a cusp at the point x, where the two paths parallel to the different velocity vectors intersect. The presence of the cusp leads to non-trivial ultra-violet behavior (for $v \neq v'$), which is described by a cusp anomalous dimension $\Gamma_c(v \cdot v')$ that was calculated at two-loop order in [20]. It coincides with the velocity-dependent anomalous dimension of heavy-quark currents, which was introduced in the context of HQET in [21]. The interpretation of heavy quarks as Wilson lines is a useful tool, which was put forward in one of the very first papers on the subject [4]. This technology will be useful in the study of the interactions of heavy quarks with collinear degrees of freedom discussed later in this review.

17.2.5. ***Heavy-quark expansion for inclusive decays*** : The theoretical description of inclusive decays of hadrons containing a heavy quark exploits two observations [22–26]: bound-state effects related to the initial state can be calculated using the heavy-quark expansion, and the fact that the final state consists of a sum over many hadronic channels eliminates the sensitivity to the properties of individual final-state hadrons. The second feature rests on the hypothesis of quark-hadron duality, i.e. the assumption that decay rates are calculable in QCD after a smearing procedure has been applied [27]. In semileptonic decays, the integration over the lepton spectrum provides a smearing over the invariant hadronic mass of the final state (global duality). For nonleptonic decays, where the total hadronic mass is fixed, the summation over many hadronic final states provides an averaging (local duality). Since global duality is a much weaker assumption, the theoretical control of inclusive semileptonic decays is on firmer footing.

Using the optical theorem, the inclusive decay width of a hadron H_b containing a b quark can be written in the form

$$\Gamma(H_b) = \frac{1}{M_{H_b}}\,\mathrm{Im}\,\langle H_b|\, i\int \mathrm{d}^4x\, T\left\{\mathcal{H}_{\mathrm{eff}}(x), \mathcal{H}_{\mathrm{eff}}(0)\right\}|H_b\rangle\,. \tag{17.6}$$

The effective weak Hamiltonian for b-quark decays consists of dimension-6 four-fermion operators and dipole operators [28]. Because of the large mass of the b quark, it follows that the separation of fields in the time-ordered product in Eq. (17.6) is small, of order $x \sim 1/m_b$. It is thus possible to construct an operator-product expansion (OPE) for the time-ordered product, in which it is represented as a series of local operators in HQET. The leading operator $\bar{h}_v h_v$ has a trivial matrix element. The next contributions arise at $\mathcal{O}(1/m_b^2)$ and give rise to two parameters $\mu_\pi^2(H_b)$ and $\mu_G^2(H_b)$, which are defined as the matrix elements of the heavy-quark kinetic energy and chromo-magnetic interaction inside the hadron H_b, respectively [29]. For the ground-state heavy mesons and baryons, one has $\mu_G^2(B) = 3(m_{B^*}^2 - m_B^2)/4 \simeq 0.36\,\mathrm{GeV}^2$ and $\mu_G^2(\Lambda_b) = 0$. Thus, the total inclusive decay rate of a hadron H_b can be written as [23,24]

$$\Gamma(H_b) = \frac{G_F^2 m_b^5 |V_{cb}|^2}{192\pi^3}\,[c_1 + c_2\,\frac{\mu_\pi^2(H_b)}{2m_b^2} + c_3\,\frac{\mu_G^2(H_b)}{2m_b^2} + \mathcal{O}(\frac{1}{m_b^3}) + \ldots], \tag{17.7}$$

where the prefactor arises from the loop integrations and is proportional to the fifth power of the b-quark mass. The coefficient functions c_i are calculable order by order in perturbation theory.

From the fully inclusive width in Eq. (17.7) one can obtain the lifetime of a heavy hadron via $\tau(H_b) = 1/\Gamma(H_b)$. Due to the universality of the leading term in the heavy-quark expansion, lifetime ratios such as $\tau(B^-)/\tau(\bar{B}^0)$, $\tau(\bar{B}_s^0)/\tau(\bar{B}^0)$ and $\tau(\Lambda_b)/\tau(\bar{B}^0)$ are particularly sensitive to the hadronic parameters determining the power corrections in the expansion. In order to understand these ratios theoretically, it is necessary to include phase-space enhanced power corrections of order $(\Lambda_{\rm QCD}/m_b)^3$ [30,31] as well as short-distance perturbative effects [32] in the calculation.

A formula analogous to Eq. (17.7) can be derived for differential distributions in specific inclusive decay processes, assuming that these distributions are integrated over a sufficiently large region of phase space to ensure quark-hadron duality. Important examples are the distributions in the lepton energy and the lepton invariant mass, as well as moments of the invariant hadronic mass distribution in the semileptonic processes $\bar{B} \to X_u \ell \bar{\nu}$ and $\bar{B} \to X_c \ell \bar{\nu}$. A global fit of semileptonic decay distributions can be used to determine the CKM matrix elements $|V_{ub}|$ and $|V_{cb}|$ along with heavy-quark parameters such as the masses m_b, m_c and the hadronic parameters $\mu_\pi^2(B)$, $\mu_G^2(B)$. These determinations provide some of the most accurate values for these parameters [33].

17.2.6. ***Shape functions and non-local power corrections*** : In certain regions of phase space, in which the hadronic final state in an inclusive heavy-hadron decay is made up of light energetic partons, the local OPE for inclusive decays must be replaced by a more complicated expansion involving hadronic matrix elements of non-local light-ray operators [34,35]. Prominent examples are the radiative decay $\bar{B} \to X_s\gamma$ for large photon energy E_γ near $m_B/2$, and the semileptonic decay $\bar{B} \to X_u \ell \bar{\nu}$ at large lepton energy or small hadronic invariant mass. In these cases, the differential decay rates at leading order in the heavy-quark expansion can be written in the factorized form $d\Gamma = H\, J \otimes S$ [36], where the hard function H and the jet function J are calculable in perturbation theory. The characteristic scales for these functions are set by m_b and $(m_b\Lambda_{\rm QCD})^{1/2}$, respectively. The soft function

$$S(\omega) = \int \frac{dt}{4\pi}\, e^{-i\omega t}\, \langle \bar{B}(v)|\, \bar{h}_v(tn) Y_n(tn) Y_n^\dagger(0) h_v(0) |\bar{B}(v)\rangle \tag{17.8}$$

is a genuinely non-perturbative object called the shape function [34,35]. Here Y_n are soft Wilson lines along a light-like direction n aligned with the momentum of the hadronic final-state jet. The jet function and the shape function share a common variable $\omega \sim \Lambda_{\rm QCD}$, and the symbol $\otimes$ denotes a convolution in this variable.

While the hard functions are different for the decays $\bar{B} \to X_s\gamma$ and $\bar{B} \to X_u \ell\bar{\nu}$, the jet and soft functions are identical at leading order in $\Lambda_{\rm QCD}/m_Q$. This is particularly important for the shape function, which introduces non-perturbative physics into the theoretical predictions for the decay rates in the regions of experimental interest. The fact that both processes depend on the same non-perturbative function makes it possible to use the measured shape of the $\bar{B} \to X_s\gamma$ photon spectrum to reduce the theoretical uncertainties in the determination of the CKM element $|V_{ub}|$ from semileptonic decays. In higher orders of the heavy-quark expansion, an increasing number of subleading jet and soft functions are required to describe the decay distributions [37]. These have been analyzed in detail at order $1/m_b$ [38–40]. In the case of $\bar{B} \to X_s\gamma$, some of these non-local effects survive in the total decay rate and give rise to irreducible hadronic uncertainties [41]. The technology for deriving the corresponding factorization theorems relies on the soft-collinear effective theory, to which we now turn.

17.3. Soft-Collinear Effective Theory

As discussed in the previous section, soft gluons that bind a heavy quark inside a heavy meson cannot change the virtuality of that heavy quark by a significant amount. The ratio $\Lambda_{\rm QCD}/m_Q$ provides the expansion parameter in HQET, which is a small parameter since $m_Q \gg \Lambda_{\rm QCD}$. This obviously does not work when considering light quarks. However, if the energy Q of the quarks is large, the ratio $\Lambda_{\rm QCD}/Q$ provides a small parameter, which can be used to construct an effective theory. One major difference to HQET is that light energetic quarks cannot only emit soft gluons, but they can also emit collinear gluons (an energetic gluon in the same direction as the original quark), without parametrically changing their virtuality. Thus, to fully reproduce the long-distance physics of energetic quarks requires that one includes their interactions with both soft and collinear particles. The resulting effective theory is therefore called soft-collinear effective theory (SCET) [42–44].

A single energetic particle can always be boosted to a frame where all momentum components have similar size, in which case there is no small expansion parameter. Thus the presence of energetic particles must refer to a reference frame defined by external kinematics. SCET has a wide range of applications; some examples are the production of energetic, light states in the decay of a heavy particle in its rest frame, the production of energetic jets in collider environments, and the scattering of energetic particles off a target at rest. In this brief review we will outline the main features of this effective theory and mention a few selected applications.

17.3.1. ***General idea of the expansion*** : Consider a quark with virtuality much less than its energy Q, moving along the direction $\vec{n}$. It is convenient to parameterize the momentum p_n of this particle in terms of its light-cone components, defined by $(p_n^-, p_n^+, p_n^\perp) = (\bar{n}\cdot p_n, n\cdot p_n, p_n^\perp)$, where $n^\mu = (1, \vec{n})$ and $\bar{n}^\mu = (1, -\vec{n})$ are light-like vectors, and $n\cdot p_n^\perp = \bar{n}\cdot p_n^\perp = 0$. The subscript n on the momentum indicates the direction of the collinear particle. In terms of these light-cone components, the virtuality satisfies $p_n^2 = p_n^+ p_n^- + p_n^{\perp 2}$. The individual components of the momentum obey

$$(p_n^-, p_n^+, p_n^\perp) \sim Q(1, \lambda^2, \lambda), \tag{17.9}$$

where $\lambda^2 = p^2/Q^2$ is the expansion parameter of SCET. The virtuality of such an energetic particle remains parametrically unchanged if it interacts with energetic particles in the same direction n, or with soft particles with momentum scaling as

$$(p_s^-, p_s^+, p_s^\perp) \sim Q(\lambda^2, \lambda^2, \lambda^2). \tag{17.10}$$

SCET is constructed in such a way as to reproduce the long-distance dynamics arising from the interactions of collinear and soft degrees of freedom.

In the above power counting the transverse momenta of soft degrees of freedom scale as $p_s^\perp \sim Q\lambda^2$, which is much smaller than the transverse momenta $p_c^\perp \sim Q\lambda$ of collinear fields. This theory is usually called SCET$_\text{I}$. If the external kinematics require that the transverse momenta of both soft and collinear fields are of the same size, $p_c^\perp \sim p_s^\perp$, then the appropriate degrees of freedom have the scaling $p_c \sim Q(1,\lambda^2,\lambda)$ and $p_s \sim Q(\lambda,\lambda,\lambda)$. This theory is usually called SCET$_\text{II}$ and is required, e.g., for exclusive hadronic decays such as $\bar{B} \to D\pi$, where the virtuality of both collinear and soft degrees of freedom are set by Λ_QCD, or for the description of transverse-momentum distributions at colliders.

17.3.2. ***Leading-order Lagrangian*** : The derivation of the SCET Lagrangian follows similar steps as described for HQET in Section 17.2.1. One begins by deriving the Lagrangian for a theory containing only a single collinear sector. Similar to HQET, one separates the full QCD field into two components, $q_n(x) = \psi_n(x) + \Xi_n(x)$, where (with $n \cdot \bar{n} = 2$)

$$\psi_n(x) = \frac{\not{n}\not{\bar{n}}}{4}\, q_n(x)\,, \qquad \Xi_n(x) = \frac{\not{\bar{n}}\not{n}}{4}\, q_n(x). \tag{17.11}$$

The degrees of freedom described by the field Ξ_n are far off shell and can therefore be eliminated using its equation of motion. This gives

$$\mathcal{L}_n = \bar{\psi}_n(x)\left[in\cdot D + i\not{D}^\perp \frac{1}{i\bar{n}\cdot D} i\not{D}^\perp\right]\frac{\not{\bar{n}}}{2}\,\psi_n(x). \tag{17.12}$$

As a next step, one separates the large and residual momentum components by decomposing the collinear momentum into a “label” and a residual momentum, $p^\mu = P^\mu + k^\mu$ with $n \cdot P = 0$. One then performs a phase redefinition on the collinear fields, such that $\psi_n(x) = e^{iP\cdot x}\,\xi_n(x)$. Derivatives acting on the fields $\xi_n(x)$ now only pick out the residual momentum. Since unlike in HQET the label momentum in SCET is not conserved, one defines a label operator $\mathcal{P}^\mu$ acting as $\mathcal{P}^\mu\xi_n(x) = P^\mu\xi_n(x)$ [43], as well as a corresponding covariant label operator $i\mathcal{D}_n^\mu = \mathcal{P}^\mu + gA_n^\mu(x)$. Note that at leading order in power counting $i\mathcal{D}_n^\mu$ does not contain the soft gluon field. This leads to the final SCET Lagrangian [43–46]

$$\mathcal{L}_n = \bar{\xi}_n(x)\left[in\cdot D_n + gn\cdot A_s + i\not{\mathcal{D}}_n^\perp \frac{1}{i\bar{n}\cdot \mathcal{D}_n} i\not{\mathcal{D}}_n^\perp\right]\frac{\not{\bar{n}}}{2}\,\xi_n(x) + \ldots, \tag{17.13}$$

where we have split $in \cdot D$ into a collinear piece $in\cdot D_n = in\cdot\partial + gn\cdot A_n$ and a soft piece $gn \cdot A_s$. This latter term gives rise to the only interaction between a collinear quark and soft gluons at leading power in λ. The ellipses represent higher-order interactions between soft and collinear particles.

The Lagrangian describing collinear fields in different light-like directions is simply given by the sum of the Lagrangians for each direction n, i.e. $\mathcal{L} = \sum_n \mathcal{L}_n$. The soft gluons are the same in each individual Lagrangian. An alternative way to understand the separation between large and small momentum components is to derive the Lagrangian of SCET in position space [46]. In this case no label operators are required, and the dependence on short-distance effects is contained in non-localities at short distances. An important difference between SCET and HQET is that the SCET Lagrangian is not corrected by short distance fluctuations. The physical reason is that in the construction described above no high-momentum modes have been integrated out [46]. Such hard modes arise when different collinear sectors are coupled via some external current (e.g. in jet production at e^+e^- or hadron colliders), or when collinear particles are produced in the rest frame of a decaying heavy object (such as in B decays). Short-distance effects are then incorporated in the Wilson coefficients of the external source operators.

17.3.3. ***Collinear gauge invariance and Wilson lines*** : An important aspect of SCET is the implementation of local gauge invariance. Because the effective field operators describe modes with certain momentum scalings, the effective Lagrangian respects only residual gauge symmetries. One of them satisfies the collinear scaling

$$(\bar{n}\cdot\partial, n\cdot\partial, \partial^{\perp})\, U_n(x) \sim Q(1,\lambda^2,\lambda)\, U_n(x), \tag{17.14}$$

and one the soft scaling

$$(\bar{n}\cdot\partial, n\cdot\partial, \partial^{\perp})\, U_s(x) \sim Q(\lambda^2,\lambda^2,\lambda^2)\, U_s(x). \tag{17.15}$$

The fact that collinear fields in different directions do not transform under the same gauge transformations implies that each collinear sector, containing particles with large momenta along a certain direction, has to be separately gauge invariant. This requires the introduction of collinear Wilson lines [43]

$$W_n(x) = P\exp\left[-ig\int_{-\infty}^{0} ds\,\bar{n}\cdot A_n(s\bar{n}+x)\right], \tag{17.16}$$

which transform under collinear gauge transformations according to $W_n \to U_n W_n$. Thus, the combination $\chi_n \equiv W_n^\dagger\,\psi_n$ is gauge invariant. In a similar manner, one can define the gauge-invariant gluon field $B_n^\mu = g^{-1} W_n^\dagger\, iD_n^\mu W_n$ [47,48]. Collinear operators in SCET are typically constructed from such gauge-invariant building blocks.

17.3.4. ***Derivation of factorization theorems*** : One of the important applications of SCET is to understand how to factorize cross sections involving energetic particles moving in different directions into simpler pieces that can either be calculated perturbatively or determined from data. Factorization theorems have been around for much longer than SCET; see [49] for a review. However, the effective theory allows for a conceptually simpler understanding of certain classes of factorization theorems [47], since most simplifications happen already at the level of the Lagrangian. The discussion in this section is valid to leading order in the power counting of the effective theory.

As discussed in the previous section, the Lagrangian of SCET does not involve any couplings between collinear particles moving in different directions. Soft gluons couple to collinear quarks only through the term $\bar{\xi}_n\, g\, n\cdot A_s(\not{\bar{n}}/2)\,\xi_n$ in the effective Lagrangian in Eq. (17.13). This coupling is similar to the coupling of soft gluons to heavy quarks in HQET, see Section 17.2.4. It can be removed by means of the field redefinition [44]

$$\psi_n(x) = Y_n(x)\psi_n^{(0)}(x)\,, \qquad A_n^a(x) = Y_n^{ab}(x)A_n^{b(0)}(x), \tag{17.17}$$

where Y_n and Y_n^{ab} live in the fundamental and adjoint representations of SU(3), respectively. This fact greatly facilitates proofs of factorization theorems in SCET. A QCD operator $O(x)$ describing the interactions of collinear partons moving in different directions can thus be written as (omitting color indices for simplicity)

$$\langle O(x)\rangle = C_O(\mu)\, \langle \mathcal{C}_{n_a}^{(0)}(x)\mathcal{C}_{n_b}^{(0)}(x)\mathcal{C}_{n_1}^{(0)}(x)\ldots\mathcal{C}_{n_N}^{(0)}(x)[\mathcal{Y}_{n_a}\mathcal{Y}_{n_b}\mathcal{Y}_{n_1}\ldots\mathcal{Y}_{n_N}](x)\rangle_\mu. \tag{17.18}$$

Here $\mathcal{C}_{n_i}^{(0)}(x)$ denotes a gauge-invariant combination of collinear fields (either quark or gluon fields) in the direction n_i. The hard matching coefficient C_O accounts for short-distance effects at the scale Q. The soft Wilson lines can either be in a color triplet or color octet representation, and are collectively denoted by $\mathcal{Y}_{n_i}$. Both the matrix elements and the coefficient C_O depend on the renormalization scale μ.

Having defined the operator mediating a given process, one can calculate the cross section by squaring the operator, taking the forward matrix element and integrating over the phase space of all final-state particles. The absence of interactions between collinear degrees of freedom moving along different directions or soft degrees of freedom implies that the forward matrix element can be factorized as

$$\begin{aligned}\big\langle \mathrm{in}\big|O(x)O^\dagger(0)\big|\mathrm{in}\big\rangle &= |C_0(\mu)|^2\, \big\langle \mathrm{in}_a\big|\mathcal{C}_{n_a}(x)\mathcal{C}_{n_a}^\dagger(0)\big|\mathrm{in}_a\big\rangle_\mu \big\langle \mathrm{in}_b\big|\mathcal{C}_{n_b}(x)\mathcal{C}_{n_b}^\dagger(0)\big|\mathrm{in}_b\big\rangle_\mu \\ &\quad \times \big\langle 0\big|\mathcal{C}_{n_1}(x)\mathcal{C}_{n_1}^\dagger(0)\big|0\big\rangle_\mu \cdots \big\langle 0\big|\mathcal{C}_{n_N}(x)\mathcal{C}_{n_N}^\dagger(0)\big|0\big\rangle_\mu \\ &\quad \times \big\langle 0\big|[\mathcal{Y}_{n_a}\ldots\mathcal{Y}_{n_N}](x)[\mathcal{Y}_{n_a}\ldots\mathcal{Y}_{n_N}]^\dagger(0)\big|0\big\rangle_\mu.\end{aligned} \tag{17.19}$$

Thus, the matrix element can be written as a product of simpler structures, each of which can be evaluated separately.

The vacuum matrix elements of the outgoing collinear fields are determined by jet functions $J_i(\mu)$. As long as the relevant scale (for example the jet mass) is sufficiently large, these functions can be calculated perturbatively. The matrix elements of the incoming collinear fields are non-perturbative objects $B_{p/N}(\mu)$ called beam functions for parton p in nucleon N [50]. For many applications they can be related perturbatively to the well-known parton distribution functions. Finally, the vacuum matrix element of the soft Wilson lines defines a so-called soft function $S_{ab\ldots N}(\mu)$. The shared dependence on x in the above equation implies that in momentum space the various components of the factorization theorem are convoluted with one another. Deriving this convolution requires a careful treatment of the phase-space integration, in particular treating the large and residual components of each momentum appropriately.

Putting all information together, the differential cross section for a proton-proton collision with N jet-like objects can schematically be written as

$$\mathrm{d}\sigma \sim \sum_{ab} H_{ab}(\mu)[B_{a/P}(\mu)B_{b/P}(\mu)] \otimes [J_1(\mu)\ldots J_N(\mu)] \otimes S_{ab\ldots N}(\mu). \tag{17.20}$$

The hard function is equal to the square of the matching coefficient, $H_{ab}(\mu) = |C_O(\mu)|^2$. It should be mentioned that the most difficult part of traditional factorization proofs involves showing that so-called Glauber gluons do not spoil the above factorization theorem [51]. This question has not yet been fully addressed in the context of SCET.

17.3.5. ***Resummation of large logarithms*** : SCET can be used to sum the large logarithms arising in perturbative calculations to all orders in the strong coupling constant α_s. In general, perturbation theory will generate a logarithmic dependence on any ratio of scales r in a problem. For processes that involve initial or final states with energy much in excess of their mass, there are two powers of logarithms for every power of α_s. These are referred to as Sudakov logarithms. For widely separated scales these large logarithms can spoil the convergence of fixed-order perturbation theory. One thus needs to reorganize the expansion in such a way that $\alpha_s L = \mathcal{O}(1)$ is kept fixed, with $L = \ln r$. More precisely, a proper resummation requires summing logarithms of the form $\alpha_s^n L^m$ with $m \leq n+1$ in the logarithm of a cross section, by writing $\ln \sigma \sim L g_0(\alpha_s L) + g_1(\alpha_s L) + \alpha_s g_2(\alpha_s L) + \ldots$, with functions $g_n(x)$ that need to be determined.

The important ingredient in achieving this resummation is the fact that SCET factorizes a given cross section into simpler pieces, each of which depends on a single physical scale. The only dependence on that scale can arise through logarithms of its ratio with the renormalization scale μ. Thus, for each of the components in the factorization theorem one can choose a renormalization scale μ for which the large logarithmic terms are absent. Of course, the factorization formula requires a common renormalization scale μ in all its components, and one therefore has to use the renormalization group (RG) to evolve the various component functions from their preferred scale to the common scale μ. A novel feature of RG equations in SCET, as opposed to other EFTs, is that the anomalous dimensions entering the evolution equations of the hard, beam, jet and soft functions in a factorization formula such as Eq. (17.20) contain a single power of the logarithm of the relevant energy scale. For example, the anomalous dimension γ_H of the hard function has the form

$$\gamma_H(\mu) = c_H \, \Gamma_{\rm cusp}(\alpha_s) \, \ln \frac{Q^2}{\mu^2} + \gamma(\alpha_s), \tag{17.21}$$

where c_H is a process-dependent coefficient and $\Gamma_{\rm cusp}$ denotes the so-called cusp anomalous dimension [20,52]. The non-cusp part γ of the anomalous dimension is process dependent. The presence of a logarithm in the anomalous dimension is characteristic of Sudakov problems and arises since the perturbative series contains double logarithms of scale ratios.

The anomalous dimension γ_H is known at two-loop order for arbitrary n-parton amplitudes containing massless or massive external partons [53–56]. Solving the RG equations one can systematically resum all large logarithms of scale ratios in the factorized cross section and express the functions $g_n(\alpha_s L)$ introduced above in terms of ratios of running coupling constants. In order to compute the first two terms $L g_0(\alpha_s L) + g_1(\alpha_s L)$ in $\ln \sigma$, corresponding to the next-to-leading logarithmic (NLL) approximation, one needs two-loop expressions for the cusp anomalous dimension and β function, one-loop expressions for the non-cusp pieces in the anomalous dimensions, and tree-level matching conditions for all component functions at their characteristic scales. To calculate the next term $\alpha_s g_2(\alpha_s L)$ in the expansion, corresponding to NNLL order, one needs to go one order higher in the loop expansion, and so on.

17.3.6. ***Factorization and resummation in*** $\mathbf{SCET_{II}}$: The effective theory $\mathrm{SCET_{II}}$ contains collinear and soft particles with momenta scaling as $(p_n^-, p_n^+, p_n^\perp) \sim Q(1, \lambda^2, \lambda)$ and $(p_s^-, p_s^+, p_s^\perp) \sim Q(\lambda, \lambda, \lambda)$. They have the same small virtuality ($p_n^2 \sim p_s^2 \sim Q^2\lambda^2$) but differ in their rapidities. An important class of observables, for which this scaling is relevant, contains cross sections for processes in which the transverse momenta of particles are constrained by external kinematics. The prime example are the transverse-momentum distributions of electroweak gauge bosons or Higgs bosons produced at hadron colliders. The parton transverse momenta are constrained by the fact that their vector sum must be equal and opposite to the transverse momentum q_T of the boson. Standard RG evolution in the effective theory controls the logarithms arising from the fact that the virtualities of the collinear and soft modes are much smaller than the hard scale Q in the process (the boson mass). However, additional large logarithms arise since the rapidities of collinear and soft modes are parametrically different, such that $e^{|y_c - y_s|} \sim 1/\lambda$. These logarithms need to be factorized in the cross section and resummed by other means.

Two equivalent approaches exist for how to deal with the additional rapidity logarithms. In the first approach, they are interpreted as a consequence of a "collinear anomaly" of the effective theory $\mathrm{SCET_{II}}$, resulting from the fact that a classical rescaling symmetry of the effective Lagrangian is broken by quantum effects [57]. The extra large logarithms can be resummed by means of simple differential equations, which typically state that (in an appropriate space) the logarithm of the cross section contains only a single logarithm of $\lambda \sim q_T/Q$, to all orders in perturbation theory. An alternative approach to resum the rapidity logarithms uses the "rapidity renormalization group", in which the relevant differential equations are obtained by considering a new type of scale variation in a parameter ν, which separates the phase space for collinear and soft particles along a hyperbola in the (p_-, p_+) plane [58]. In contrast to the standard RG, there is no running coupling involved in the ν evolution, since the different contributions live at the same virtuality.

$\mathrm{SCET_{II}}$ also plays an important role in the study of factorization for a variety of exclusive B meson decays, such as $\bar{B} \to \pi\ell\nu$, $\bar{B} \to K^*\gamma$ and $\bar{B} \to \pi\pi$, for which the virtualities of energetic (collinear) final-state particles are of order Λ_{QCD}, which is also the scale for the soft light degrees of freedom contained in the initial-state B meson.

17.3.7. ***Applications*** : Most of the applications of SCET are either in flavor physics, where the decay of a heavy B meson can give rise to energetic light partons, or in collider physics, where the presence of jets naturally leads to collimated sets of energetic particles. For many of these applications alternative approaches existed before the invention of SCET, but the effective theory has opened up alternative ways to understand the physics of these processes. For several examples, however, SCET has allowed new insights. The investigation of heavy-to-light form factors has been instrumental for understanding factorization in exclusive semileptonic B decays [59]. SCET has also provided a field-theoretic basis for the QCD factorization approach to exclusive, non-leptonic decays of B mesons [60]. Using SCET methods, proofs of factorization were derived for the color-allowed decay $\bar{B}^0 \to D^+\pi^-$ [61], the color-suppressed decay $\bar{B}^0 \to D^0\pi^0$ [62], and the radiative decay $\bar{B} \to K^*\gamma$ [63]. Further examples are factorization theorems and the resummation of endpoint logarithms for quarkonia production [64], the resummation of large logarithmic terms for the thrust [65] and jet broadening [66] distributions in e^+e^- annihilation beyond NLL order, the development of new factorizable observables to veto extra jets [67], all-orders factorization theorems for processes containing electroweak Sudakov logarithms [68], and the resummation of threshold (soft gluon) logarithms for several important processes at hadron colliders [69–71]. Recently, there has been a lot of activity describing p_T-based resummation at hadron colliders. Examples are the transverse-momentum distributions of electroweak bosons [57] and jets [72]. We now describe three applications in more detail.

Event-shape distributions, in particular the thrust distribution, have been measured to high accuracy at LEP [73]. They can be used for a determination of the strong coupling constant α_s. SCET has increased the theoretical accuracy in the calculations of the thrust and C-parameter distributions significantly. First, it has allowed to increase the perturbative accuracy of the thrust spectrum. The resummation of logarithms of τ, which become important for $\tau \ll 1$, has been performed to N^{3}LL [65], two orders beyond what was previously available. Combining this resummation with the known two-loop spectrum [74,75] gives precise perturbative predictions both at small and large values of τ. Second, the factorization of the cross section in SCET has made it possible to include non-perturbative physics through a shape function, in analogy with the B-physics case discussed in Section 17.2.6. Comparing the theoretical predictions to the measured thrust and C-parameter distributions yields a precise value of the strong coupling constant $\alpha_s(m_Z)$, which however is lower than the average value cited in Chap. 9 [76] by several standard deviations [77,78].

The Higgs-boson production cross section in gluon fusion at the LHC, defined with a jet veto stating that no jet in the final state has transverse momentum above a threshold $p_T^{\rm veto}$, can be factorized in the form [79,80] (see [81] for a corresponding calculation outside the SCET framework)

$$\begin{aligned}\sigma(p_T^{\rm veto}) = H(m_H,\mu)\Big(\frac{\nu_B}{\nu_S}\Big)^{-2F_{gg}(R,p_T^{\rm veto},\mu)} S_{gg}\Big(R,p_T^{\rm veto},\mu,\frac{\nu_S}{p_T^{\rm veto}}\Big) \\ \times \int_\tau^1 \frac{dz}{z}\, B_{g/P}\Big(z,R,p_T^{\rm veto},\mu,\frac{\nu_B}{m_H}\Big)\, B_{g/P}\Big(\frac{\tau}{z},R,p_T^{\rm veto},\mu,\frac{\nu_B}{m_H}\Big),\end{aligned} \tag{17.22}$$

where $\tau = m_H^2/s$, and $\mu \sim p_T^{\rm veto}$ is a common factorization scale. The beam functions $B_{g/P}$, the soft function S_{gg} and the exponent F_{gg} all depend on the jet radius R as well as the jet clustering algorithm. The scale dependence of the hard function H is controlled by standard RG evolution in SCET. The beam functions can be factorized further into calculable collinear kernels convoluted with parton distribution functions. In addition to the renormalization scale μ, the beam and soft functions depend on two rapidity scales $\nu_B \sim m_H$ and $\nu_S \sim p_T^{\rm veto}$, respectively. In [79] the default values $\nu_B = m_H$ and $\nu_S = p_T^{\rm veto}$ are used for these scales, and the soft function S_{gg} is absorbed into the beam functions. In [80] the exponent F_{gg} is called $-\gamma_\nu^g/2$. The second factor on the right-hand side of the factorization formula Eq. (17.22), which resums large rapidity logarithms, implies that the logarithm of the jet-veto cross section contains a single large logarithm

$\ln\sigma = -2F_{gg}(R, p_T^{\text{veto}}, \mu)\,\ln(m_H/p_T^{\text{veto}}) + \ldots$ not contained in the hard function. Its coefficient can be calculated in fixed-order perturbation theory.

Obtaining more precise fixed-order calculations has been an important goal for many years. A major difficulty in these calculations is the proper handling of the infrared singularities that arise in both virtual and real contributions. Recently, a proposal has been made to use a so-called N-jettiness ($\mathcal{T}_N$) subtraction/slicing method to obtain the NNLO result from a much easier NLO calculation, combined with information about the singular dependence of the cross section on the $\mathcal{T}_N$ resolution variable [82,83]. While the NLO calculations can be performed using well established techniques, the singular dependence on $\mathcal{T}_N$ can be calculated using SCET at NNLO.

17.4. Open issues and perspectives

HQET has successfully passed many experimental tests, and there are not many open questions that still need to be addressed. One concept that has not been derived from first principles is the notion of quark-hadron duality, which underlies the application of HQET to the description of inclusive decays of B mesons. The validity of global duality (at energies even lower than those relevant in B decays) has been tested experimentally using high-precision data on semileptonic B decays and on hadronic τ decays, and good agreement between theory and data was found. However, assigning a theoretical uncertainty due to possible duality violations remains a difficult task. Another known issue is the that the measured values of the CKM elements $|V_{cb}|$ and $|V_{ub}|$ extracted from exclusive or inclusive decays of B mesons differ from each other by several standard deviations (see Ref. 84). Both measurements rely on the heavy-quark limit, and the uncertainties quoted include theoretical estimates of the effects of power corrections arising from the finite b-quark mass. It remains an open question whether the discrepancies are due to underestimated theoretical or experimental uncertainties, or whether they may hint to the existence of new physics.

SCET, on the other hand, is still an active field of research, and new results are being obtained regularly. An active area of research is the understanding of non-global logarithms arising in hadron-collider processes with jets [85,86]. SCET-based fixed-order calculations have helped to shed some light on the nature of these logarithms [87–89]. Another active field concerns the study of Glauber gluons in SCET [90] and their relation to the BFKL equation familiar from small-x physics [91]. A solid understanding of these issues will be necessary to make factorization proofs in SCET more rigorous. Glauber gluons also play an important role in SCET-based analysis of jet propagation in dense QCD media [92–95], which gives rise to the jet-quenching phenomenon in heavy-ion collisions. An important open questions facing some applications of SCET concerns factorized expressions containing endpoint-divergent convolution integrals. This problem arises, for example, in the description of heavy-to-light form factors such as $F_{\bar{B}\to\pi}(q^2)$ at large recoil [96].

We close this short review by mentioning a particularly nice application combining the methods of heavy-particle EFTs such as HQET and non-relativistic QCD with SCET in the context of describing the interactions of heavy dark matter (with mass $M \gg v$) with

SM particles. In [97] is was realized that the interactions of heavy, weakly interacting massive particles (WIMPs) with nuclear targets can be described in a model-independent way using heavy-particle EFTs. The WIMPs are charged under $\mathrm{SU}(2)_L$ and can interact with electroweak gauge bosons and the Higgs boson. The WIMP EFT was later extended by describing the produced, highly energetic electroweak gauge bosons in terms of soft or collinear fields in SCET [98–100]. This allows one to systematically separate all relevant mass scales, resum electroweak Sudakov logarithms and disentangle the so-called Sommerfeld enhancement from the short-distance hard annihilation process.

References:

1. E. Witten, Nucl. Phys. **B122**, 109 (1977).
2. S. Weinberg, Phys. Lett. **B91**, 51 (1980).
3. L.J. Hall, Nucl. Phys. **B178**, 75 (1981).
4. E. Eichten, B. Hill, Phys. Lett. **B234**, 511 (1990).
5. H. Georgi, Phys. Lett. **B240**, 447 (1990).
6. B. Grinstein, Nucl. Phys. **B339**, 253 (1990).
7. T. Mannel, W. Roberts, Z. Ryzak, Nucl. Phys. **B368**, 204 (1992).
8. M. Neubert, Phys. Reports **245**, 259 (1994).
9. A.V. Manohar, M.B. Wise, Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. **10**, 1 (2000).
10. M.E. Luke, A.V. Manohar, Phys. Lett. **B286**, 348 (1992).
11. E.V. Shuryak, Phys. Lett. **B93**, 134 (1980).
12. N. Isgur, M.B. Wise, Phys. Lett. **B232**, 113 (1989); Phys. Lett. **B237**, 527 (1990).
13. N. Isgur, M.B. Wise, Phys. Rev. Lett. **66**, 1130 (1991).
14. S. Nussinov, W. Wetzel, Phys. Rev. **D36**, 130 (1987).
15. M.A. Shifman, M.B. Voloshin, Sov. J. Nucl. Phys. **45**, 292 (1987); Sov. J. Nucl. Phys. **47**, 511 (1988).
16. M. Neubert, Phys. Lett. **B264**, 455 (1991).
17. M.E. Luke, Phys. Lett. **B252**, 447 (1990).
18. I.I.Y. Bigi *et al.*, Phys. Rev. **D52**, 196 (1995).
19. N. Uraltsev, Phys. Lett. **B501**, 86 (2001).
20. G.P. Korchemsky, A.V. Radyushkin, Nucl. Phys. **B283**, 342 (1987).
21. A.F. Falk *et al.*, Nucl. Phys. **B343**, 1 (1990).
22. J. Chay, H. Georgi, B. Grinstein, Nucl. Phys. **B247**, 399 (1990).
23. I.I.Y. Bigi, N.G. Uraltsev, A.I. Vainshtein, Phys. Lett. **B293**, 430 (1992); I.I.Y. Bigi *et al.*, Phys. Rev. Lett. **71**, 496 (1993).
24. A.V. Manohar, M.B. Wise, Phys. Rev. **D49**, 1310 (1994).
25. T. Mannel, Nucl. Phys. **B413**, 396 (1994).
26. A.F. Falk, M.E. Luke, M.J. Savage, Phys. Rev. **D49**, 3367 (1994).
27. E.C. Poggio, H.R. Quinn, S. Weinberg, Phys. Rev. **D13**, 1958 (1976).
28. G. Buchalla, A.J. Buras, M.E. Lautenbacher, Rev. Mod. Phys. **68**, 1125 (1996).
29. A.F. Falk, M. Neubert, Phys. Rev. **D47**, 2965 (1993).
30. M. Neubert, C.T. Sachrajda, Nucl. Phys. **B483**, 339 (1997).
31. M. Beneke, G. Buchalla, I. Dunietz, Phys. Rev. **D54**, 4419 (1996).

32. M. Beneke, G. Buchalla, C. Greub, A. Lenz and U. Nierste, Phys. Lett. **B459**, 631 (1999); Nucl. Phys. **B639**, 389 (2002).
33. C.W. Bauer *et al.*, Phys. Rev. **D70**, 094017 (2004).
34. M. Neubert, Phys. Rev. **D49**, 4623 (1994).
35. I.I.Y. Bigi *et al.*, Int. J. Mod. Phys. **A9**, 2467 (1994).
36. G.P. Korchemsky, G.F. Sterman, Phys. Lett. **B340**, 96 (1994).
37. C.W. Bauer, M.E. Luke, T. Mannel, Phys. Rev. **D68**, 094001 (2003).
38. K.S.M. Lee, I.W. Stewart, Nucl. Phys. **B721**, 325 (2005).
39. S.W. Bosch, M. Neubert, G. Paz, JHEP **0411**, 073 (2004).
40. M. Beneke *et al.*, JHEP **0506**, 071 (2005).
41. M. Benzke *et al.*, JHEP **1008**, 099 (2010).
42. C.W. Bauer, S. Fleming, M.E. Luke, Phys. Rev. **D63**, 014006 (2000);;
    C.W. Bauer *et al.*, Phys. Rev. **D63**, 114020 (2001).
43. C.W. Bauer, I.W. Stewart, Phys. Lett. **B516**, 134 (2001).
44. C.W. Bauer, D. Pirjol, I.W. Stewart, Phys. Rev. **D65**, 054022 (2002).
45. J. Chay, C. Kim, Phys. Rev. **D65**, 114016 (2002).
46. M. Beneke *et al.*, Nucl. Phys. **B643**, 431 (2002).
47. C.W. Bauer *et al.*, Phys. Rev. **D66**, 014017 (2002).
48. R.J. Hill, M. Neubert, Nucl. Phys. **B657**, 229 (2003).
49. J.C. Collins, D.E. Soper, G.F. Sterman, Adv. Ser. Direct. High Energy Phys. **5**, 1 (1988).
50. I.W. Stewart, F.J. Tackmann, W.J. Waalewijn, Phys. Rev. **D81**, 094035 (2010).
51. J.C. Collins, D.E. Soper, G. Sterman, Nucl. Phys. **B261**, 104 (1985); Nucl. Phys. **B308**, 833 (1988).
52. I.A. Korchemskaya, G.P. Korchemsky, Phys. Lett. **B287**, 169 (1992).
53. T. Becher, M. Neubert, Phys. Rev. Lett. **102**, 162001 (2009); JHEP **0906**, 081 (2009).
54. E. Gardi, L. Magnea, JHEP **0903**, 079 (2009).
55. A. Mitov, G.F. Sterman, I. Sung, Phys. Rev. **D79**, 094015 (2009).
56. A. Ferroglia *et al.*, Phys. Rev. Lett. **103**, 201601 (2009).
57. T. Becher, M. Neubert, Eur. Phys. J. **C71**, 1665 (2011).
58. J.Y. Chiu *et al.*, JHEP **1205**, 084 (2012).
59. M. Beneke and T. Feldmann, Nucl. Phys. **B685**, 249 (2004).
60. M. Beneke *et al.*, Phys. Rev. Lett. **83**, 1914 (1999); Nucl. Phys. **B591**, 313 (2000).
61. C.W. Bauer, D. Pirjol, I.W. Stewart, Phys. Rev. Lett. **87**, 201806 (2001).
62. S. Mantry, D. Pirjol, I.W. Stewart, Phys. Rev. **D68**, 114009 (2003).
63. T. Becher, R.J. Hill, M. Neubert, Phys. Rev. **D72**, 094017 (2005).
64. S. Fleming, A.K. Leibovich, T. Mehen, Phys. Rev. **D68**, 094011 (2003).
65. T. Becher, M.D. Schwartz, JHEP **0807**, 034 (2008).
66. T. Becher, G. Bell, M. Neubert, Phys. Lett. **B704**, 276 (2011).
67. I.W. Stewart, F.J. Tackmann, W.J. Waalewijn, Phys. Rev. **D81**, 094035 (2010).
68. J.-y. Chiu, R. Kelley, A.V. Manohar, Phys. Rev. **D78**, 073006 (2008).
69. T. Becher, M. Neubert, G. Xu, JHEP **0807**, 030 (2008).

70. V. Ahrens *et al.*, Eur. Phys. J. **C62**, 333 (2009);;
    V. Ahrens *et al.*, JHEP **1009**, 097 (2010).
71. X. Liu, S. Mantry, F. Petriello, Phys. Rev. **D86**, 074004 (2012).
72. X. Liu, F. Petriello, Phys. Rev. **D87**, 014018 (2013).
73. For a review, see: S. Kluth, Rept. on Prog. in Phys. **69**, 1771 (2006).
74. A. Gehrmann-De Ridder *et al.*, Phys. Rev. Lett. **99**, 132002 (2007).
75. S. Weinzierl, Phys. Rev. Lett. **101**, 162001 (2008).
76. S. Bethke, G. Dissertori, and G.P. Salam, *"Quantum Chromodynamics"* in this *Review*, `pdg.lbl.gov`.
77. R. Abbate *et al.*, Phys. Rev. **D83**, 074021 (2011).
78. A.H. Hoang *et al.*, Phys. Rev. **D91**, 094018 (2015).
79. T. Becher, M. Neubert, JHEP **1207**, 108 (2012).
80. I.W. Stewart *et al.*, Phys. Rev. **D89**, 054001 (2014).
81. A. Banfi *et al.*, Phys. Rev. Lett. **109**, 202001 (2012).
82. R. Boughezal, X. Liu, F. Petriello, Phys. Rev. **D91**, 094035 (2015).
83. J. Gaunt *et al.*, JHEP **1509**, 058 (2015).
84. R. Kowalewski, T. Mannel, *"Semileptonic bottom hadron decays and the determination of V_{cb} and V_{ub}"*, in this *Review*, `pdg.lbl.gov`.
85. M. Dasgupta, G.P. Salam, Phys. Lett. **B512**, 323 (2001); JHEP **0203**, 017 (2002).
86. R.B. Appleby, M.H. Seymour, JHEP **0212**, 063 (2002).
87. R. Kelley *et al.*, Phys. Rev. **D84**, 045022 (2011); Phys. Rev. **D86**, 054017 (2012).
88. A. Hornig *et al.*, JHEP **1108**, 054 (2011).
89. A. von Manteuffel, R.M. Schabinger, H.X. Zhu, JHEP **1403**, 139 (2014).
90. C.W. Bauer, B.O. Lange, G. Ovanesyan, JHEP **1107**, 077 (2011).
91. S. Fleming, Phys. Lett. **B735**, 266 (2014).
92. A. Idilbi, A. Majumder, Phys. Rev. **D80**, 054022 (2009).
93. G. Ovanesyan, I. Vitev, JHEP **1106**, 080 (2011).
94. M. Benzke *et al.*, JHEP **1302**, 129 (2013).
95. Z.B. Kang *et al.*, Phys. Rev. Lett. **114**, 092002 (2015).
96. M. Beneke, T. Feldmann, Nucl. Phys. **B592**, 3 (2001).
97. R.J. Hill, M.P. Solon, Phys. Lett. **B707**, 539 (2012); Phys. Rev. Lett. **112**, 211602 (2014).
98. M. Bauer *et al.*, JHEP **1501**, 099 (2015);;
    R.J. Hill, M.P. Solon, Phys. Rev. **D91**, 043505 (2015).
99. M. Baumgart, I.Z. Rothstein, V. Vaidya, Phys. Rev. Lett. **114**, 211301 (2015).
100. G. Ovanesyan, T.R. Slatyer, I.W. Stewart, Phys. Rev. Lett. **114**, 211302 (2015).