45. SU(3) Isoscalar Factors and Representation Matrices

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The most commonly used SU(3) isoscalar factors, corresponding to the singlet, octet, and decuplet content of $8 \otimes 8$ and $10 \otimes 8$, are shown at the right. The notation uses particle names to identify the coefficients, so that the pattern of relative couplings may be seen at a glance. We illustrate the use of the coefficients below. See J.J de Swart, Rev. Mod. Phys. **35**, 916 (1963) for detailed explanations and phase conventions.

A $\sqrt{\ }$ *is to be understood over every integer in the matrices;* the exponent 1/2 on each matrix is a reminder of this. For example, the $\Xi \to \Omega K$ element of the $10 \to 10 \otimes 8$ matrix is $-\sqrt{6}/\sqrt{24} = -1/2$.

Intramultiplet relative decay strengths may be read directly from the matrices. For example, in decuplet → octet + octet decays, the ratio of $\Omega^* \to \Xi\overline{K}$ and $\Delta \to N\pi$ partial widths is, from the $10 \to 8 \times 8$ matrix,

$$\frac{\Gamma\left(\Omega^* \to \Xi\overline{K}\right)}{\Gamma\left(\Delta \to N\pi\right)} = \frac{12}{6} \times \text{ (phase space factors) .} \tag{45.1}$$

Including isospin Clebsch-Gordan coefficients, we obtain, *e.g.*,

$$\frac{\Gamma(\Omega^{*-} \to \Xi^0 K^-)}{\Gamma(\Delta^+ \to p\,\pi^0)} = \frac{1/2}{2/3} \times \frac{12}{6} \times p.s.f. = \frac{3}{2} \times p.s.f. \tag{45.2}$$

Partial widths for $8 \to 8 \otimes 8$ involve a linear superposition of 8_1 (symmetric) and 8_2 (antisymmetric) couplings. For example,

$$\Gamma(\Xi^* \to \Xi\pi) \sim \left(-\sqrt{\frac{9}{20}}\, g_1 + \sqrt{\frac{3}{12}}\, g_2\right)^2 . \tag{45.3}$$

The relations between g_1 and g_2 (with de Swart's normalization) and the standard D and F couplings that appear in the interaction Lagrangian,

$$\mathscr{L} = -\sqrt{2}\, D\, Tr\left(\{\overline{B}, B\}M\right) + \sqrt{2}\, F\, Tr\left([\overline{B}, B]\, M\right) \ , \tag{45.4}$$

where $[\overline{B}, B] \equiv \overline{B}B - B\overline{B}$ and $\{\overline{B}, B\} \equiv \overline{B}B + B\overline{B}$, are

$$D = \frac{\sqrt{30}}{40}\, g_1 \ , \qquad F = \frac{\sqrt{6}}{24}\, g_2 \ . \tag{45.5}$$

Thus, for example,

$$\Gamma(\Xi^* \to \Xi\pi) \sim (F - D)^2 \sim (1 - 2\alpha)^2 \ , \tag{45.6}$$

where $\alpha \equiv F/(D + F)$. (This definition of α is de Swart's. The alternative $D/(D + F)$, due to Gell-Mann, is also used.)

The generators of SU(3) transformations, λ_a ($a = 1, 8$), are 3×3 matrices that obey the following commutation and anticommutation relationships:

$$[\lambda_a,\ \lambda_b] \equiv \lambda_a\lambda_b - \lambda_b\lambda_a = 2i f_{abc}\lambda_c \tag{45.7}$$

$$\{\lambda_a,\ \lambda_b\} \equiv \lambda_a\lambda_b + \lambda_b\lambda_a = \frac{4}{3}\delta_{ab}I + 2d_{abc}\lambda_c \ , \tag{45.8}$$

where I is the 3×3 identity matrix, and δ_{ab} is the Kronecker delta symbol. The f_{abc} are odd under the permutation of any pair of indices, while the d_{abc} are even. The nonzero values are

1 → 8 ⊗ 8

$$\begin{pmatrix} \Lambda \end{pmatrix} \to \begin{pmatrix} N\overline{K} & \Sigma\pi & \Lambda\eta & \Xi K \end{pmatrix} = \frac{1}{\sqrt{8}} \begin{pmatrix} 2 & 3 & -1 & -2 \end{pmatrix}^{1/2}$$

8_1 → 8 ⊗ 8

$$\begin{pmatrix} N \\ \Sigma \\ \Lambda \\ \Xi \end{pmatrix} \to \begin{pmatrix} N\pi \;\; N\eta \;\; \Sigma K \;\; \Lambda K \\ N\overline{K} \;\; \Sigma\pi \;\; \Lambda\pi \;\; \Sigma\eta \;\; \Xi K \\ N\overline{K} \;\; \Sigma\pi \;\; \Lambda\eta \;\; \Xi K \\ \Sigma\overline{K} \;\; \Lambda\overline{K} \;\; \Xi\pi \;\; \Xi\eta \end{pmatrix} = \frac{1}{\sqrt{20}} \begin{pmatrix} 9 & -1 & -9 & -1 \\ -6 & 0 & 4 & 4 & -6 \\ 2 & -12 & -4 & -2 \\ 9 & -1 & -9 & -1 \end{pmatrix}^{1/2}$$

8_2 → 8 ⊗ 8

$$\begin{pmatrix} N \\ \Sigma \\ \Lambda \\ \Xi \end{pmatrix} \to \begin{pmatrix} N\pi \;\; N\eta \;\; \Sigma K \;\; \Lambda K \\ N\overline{K} \;\; \Sigma\pi \;\; \Lambda\pi \;\; \Sigma\eta \;\; \Xi K \\ N\overline{K} \;\; \Sigma\pi \;\; \Lambda\eta \;\; \Xi K \\ \Sigma\overline{K} \;\; \Lambda\overline{K} \;\; \Xi\pi \;\; \Xi\eta \end{pmatrix} = \frac{1}{\sqrt{12}} \begin{pmatrix} 3 & 3 & 3 & -3 \\ 2 & 8 & 0 & 0 & -2 \\ 6 & 0 & 0 & 6 \\ 3 & 3 & 3 & -3 \end{pmatrix}^{1/2}$$

10 → 8 ⊗ 8

$$\begin{pmatrix} \Delta \\ \Sigma \\ \Xi \\ \Omega \end{pmatrix} \to \begin{pmatrix} N\pi \;\; \Sigma K \\ N\overline{K} \;\; \Sigma\pi \;\; \Lambda\pi \;\; \Sigma\eta \;\; \Xi K \\ \Sigma\overline{K} \;\; \Lambda\overline{K} \;\; \Xi\pi \;\; \Xi\eta \\ \Xi\overline{K} \end{pmatrix} = \frac{1}{\sqrt{12}} \begin{pmatrix} -6 & 6 \\ -2 & 2 & -3 & 3 & 2 \\ 3 & -3 & 3 & 3 \\ 12 \end{pmatrix}^{1/2}$$

8 → 10 ⊗ 8

$$\begin{pmatrix} N \\ \Sigma \\ \Lambda \\ \Xi \end{pmatrix} \to \begin{pmatrix} \Delta\pi \;\; \Sigma K \\ \Delta\overline{K} \;\; \Sigma\pi \;\; \Sigma\eta \;\; \Xi K \\ \Sigma\pi \;\; \Xi K \\ \Sigma\overline{K} \;\; \Xi\pi \;\; \Xi\eta \;\; \Omega K \end{pmatrix} = \frac{1}{\sqrt{15}} \begin{pmatrix} -12 & 3 \\ 8 & -2 & -3 & 2 \\ -9 & 6 \\ 3 & -3 & -3 & 6 \end{pmatrix}^{1/2}$$

10 → 10 ⊗ 8

$$\begin{pmatrix} \Delta \\ \Sigma \\ \Xi \\ \Omega \end{pmatrix} \to \begin{pmatrix} \Delta\pi \;\; \Delta\eta \;\; \Sigma K \\ \Delta\overline{K} \;\; \Sigma\pi \;\; \Sigma\eta \;\; \Xi K \\ \Sigma\overline{K} \;\; \Xi\pi \;\; \Xi\eta \;\; \Omega K \\ \Xi\overline{K} \;\; \Omega\eta \end{pmatrix} = \frac{1}{\sqrt{24}} \begin{pmatrix} 15 & 3 & -6 \\ 8 & 8 & 0 & -8 \\ 12 & 3 & -3 & -6 \\ 12 & -12 \end{pmatrix}^{1/2}$$

abc	f_{abc}
123	1
147	$1/2$
156	$-1/2$
246	$1/2$
257	$1/2$
345	$1/2$
367	$-1/2$
458	$\sqrt{3}/2$
678	$\sqrt{3}/2$

abc	d_{abc}	abc	d_{abc}
118	$1/\sqrt{3}$	355	$1/2$
146	$1/2$	366	$-1/2$
157	$1/2$	377	$-1/2$
228	$1/\sqrt{3}$	448	$-1/(2\sqrt{3})$
247	$-1/2$	558	$-1/(2\sqrt{3})$
256	$1/2$	668	$-1/(2\sqrt{3})$
338	$1/\sqrt{3}$	778	$-1/(2\sqrt{3})$
344	$1/2$	888	$-1/\sqrt{3}$

The λ_a's are

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Equation (45.7) defines the Lie algebra of SU(3). A general d-dimensional representation is given by a set of $d \times d$ matrices satisfying Eq. (45.7) with the f_{abc} given above. Equation (45.8) is specific to the defining 3-dimensional representation.