46. SU(n) Multiplets and Young Diagrams

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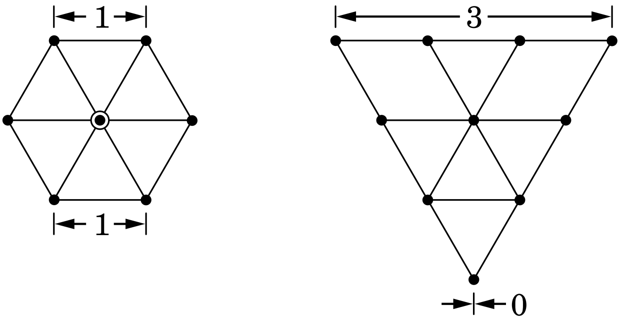
This note tells (1) how SU(n) particle multiplets are identified or labeled, (2) how to find the number of particles in a multiplet from its label, (3) how to draw the Young diagram for a multiplet, and (4) how to use Young diagrams to determine the overall multiplet structure of a composite system, such as a 3-quark or a meson-baryon system.

In much of the literature, the word "representation" is used where we use "multiplet," and "tableau" is used where we use "diagram."

46.1. Multiplet labels

An SU(n) multiplet is uniquely identified by a string of ($n-1$) nonnegative integers: $(\alpha, \beta, \gamma, \ldots)$. Any such set of integers specifies a multiplet. For an SU(2) multiplet such as an isospin multiplet, the single integer α is the number of *steps* from one end of the multiplet to the other (*i.e.*, it is one fewer than the number of particles in the multiplet). In SU(3), the two integers α and β are the numbers of steps across the top and bottom levels of the multiplet diagram. Thus the labels for the SU(3) octet and decuplet



are (1,1) and (3,0). For larger n, the interpretation of the integers in terms of the geometry of the multiplets, which exist in an ($n-1$)-dimensional space, is not so readily apparent.

The label for the SU(n) singlet is $(0, 0, \ldots, 0)$. In a flavor SU(n), the n quarks together form a $(1, 0, \ldots, 0)$ multiplet, and the n antiquarks belong to a $(0, \ldots, 0, 1)$ multiplet. These two multiplets are *conjugate* to one another, which means their labels are related by $(\alpha, \beta, \ldots) \leftrightarrow (\ldots, \beta, \alpha)$.

46.2. Number of particles

The number of particles in a multiplet, $N = N(\alpha, \beta, \ldots)$, is given as follows (note the pattern of the equations).

In SU(2), $N = N(\alpha)$ is

$$N = \frac{(\alpha+1)}{1} \,. \tag{46.1}$$

In SU(3), $N = N(\alpha, \beta)$ is

$$N = \frac{(\alpha+1)}{1} \cdot \frac{(\beta+1)}{1} \cdot \frac{(\alpha+\beta+2)}{2} \,. \tag{46.2}$$

In SU(4), $N = N(\alpha, \beta, \gamma)$ is

$$N = \frac{(\alpha{+}1)}{1} \cdot \frac{(\beta{+}1)}{1} \cdot \frac{(\gamma{+}1)}{1} \cdot \frac{(\alpha{+}\beta{+}2)}{2} \cdot \frac{(\beta{+}\gamma{+}2)}{2} \cdot \frac{(\alpha{+}\beta{+}\gamma{+}3)}{3} \,. \tag{46.3}$$

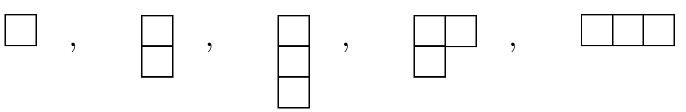
Note that in Eq. (46.3) there is no factor with $(\alpha + \gamma + 2)$: only a *consecutive* sequence of the label integers appears in any factor. One more example should make the pattern clear for any SU(n). In SU(5), $N = N(\alpha, \beta, \gamma, \delta)$ is

$$\begin{aligned} N = {} & \frac{(\alpha{+}1)}{1} \cdot \frac{(\beta{+}1)}{1} \cdot \frac{(\gamma{+}1)}{1} \cdot \frac{(\delta{+}1)}{1} \cdot \frac{(\alpha{+}\beta{+}2)}{2} \cdot \frac{(\beta{+}\gamma{+}2)}{2} \\ & \times \frac{(\gamma{+}\delta{+}2)}{2} \cdot \frac{(\alpha{+}\beta{+}\gamma{+}3)}{3} \cdot \frac{(\beta{+}\gamma{+}\delta{+}3)}{3} \cdot \frac{(\alpha{+}\beta{+}\gamma{+}\delta{+}4)}{4} \,. \end{aligned} \tag{46.4}$$

From the symmetry of these equations, it is clear that multiplets that are conjugate to one another have the same number of particles, but so can other multiplets. For example, the SU(4) multiplets (3,0,0) and (1,1,0) each have 20 particles. Try the equations and see.

46.3. Young diagrams

A Young diagram consists of an array of boxes (or some other symbol) arranged in one or more *left-justified* rows, with each row being *at least as long* as the row beneath. The correspondence between a diagram and a multiplet label is: The top row juts out α boxes to the right past the end of the second row, the second row juts out β boxes to the right past the end of the third row, *etc.* A diagram in SU(n) has at most n rows. There can be any number of "completed" columns of n boxes buttressing the left of a diagram; these don't affect the label. Thus in SU(3) the diagrams

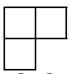
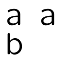


represent the multiplets (1,0), (0,1), (0,0), (1,1), and (3,0). In any SU(n), the quark multiplet is represented by a single box, the antiquark multiplet by a column of ($n{-}1$) boxes, and a singlet by a completed column of n boxes.

46.4. Coupling multiplets together

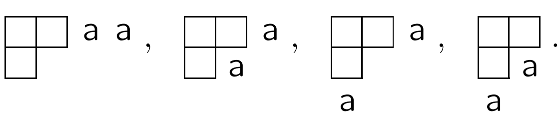
The following recipe tells how to find the multiplets that occur in coupling two multiplets together. To couple together more than two multiplets, first couple two, then couple a third with each of the multiplets obtained from the first two, *etc.*

First a definition: A sequence of the letters $a, b, c, \ldots$ is *admissible* if at any point in the sequence at least as many a's have occurred as b's, at least as many b's have occurred as c's, *etc.* Thus *abcd* and *aabcb* are admissible sequences and *abb* and *acb* are not. Now the recipe:

(a) Draw the Young diagrams for the two multiplets, but in one of the diagrams replace the boxes in the first row with a's, the boxes in the second row with b's, *etc.* Thus, to couple two SU(3) octets (such as the π-meson octet and the baryon octet), we start with  and 

. The *unlettered* diagram forms the *upper left-hand corner* of all the enlarged diagrams constructed below.

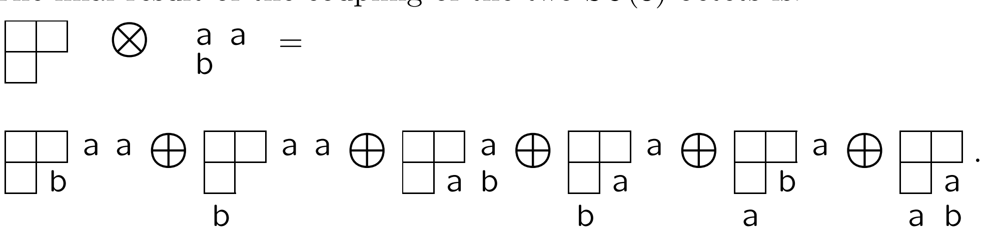
(b) Add the a's from the lettered diagram to the right-hand ends of the rows of the unlettered diagram to form all possible legitimate Young diagrams that have no more than one a per column. In general, there will be several distinct diagrams, and all the a's appear in each diagram. At this stage, for the coupling of the two SU(3) octets, we have:



(c) Use the b's to further enlarge the diagrams already obtained, subject to the same rules. Then throw away any diagram in which the full sequence of letters formed by reading *right to left* in the first row, then the second row, *etc.*, is not admissible.

(d) Proceed as in (c) with the c's (if any), *etc.*

The final result of the coupling of the two SU(3) octets is:



Here only the diagrams with admissible sequences of a's and b's and with fewer than four rows (since $n = 3$) have been kept. In terms of multiplet labels, the above may be written

$$(1,1) \otimes (1,1) = (2,2) \oplus (3,0) \oplus (0,3) \oplus (1,1) \oplus (1,1) \oplus (0,0) .$$

In terms of numbers of particles, it may be written

$$\mathbf{8} \otimes \mathbf{8} = \mathbf{27} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1} .$$

The product of the numbers on the left here is equal to the sum on the right, a useful check. (See also Sec. 15 on the Quark Model.)