76. $K^{\pm}_{\ell 3}$ and $K^0_{\ell 3}$ Form Factors

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Assuming that only the vector current contributes to $K \to \pi \ell \nu$ decays, we write the matrix element as

$$M \propto f_+(t)\left[(P_K + P_\pi)_\mu \overline{\ell}\gamma_\mu(1+\gamma_5)\nu\right] + f_-(t)\left[m_\ell \overline{\ell}(1+\gamma_5)\nu\right] \; , \tag{76.1}$$

where P_K and P_π are the four-momenta of the K and π mesons, m_ℓ is the lepton mass, and f_+ and f_- are dimensionless form factors which can depend only on $t = (P_K - P_\pi)^2$, the square of the four-momentum transfer to the leptons. If time-reversal invariance holds, f_+ and f_- are relatively real. $K_{\mu 3}$ experiments, discussed immediately below, measure f_+ and f_-, while K_{e3} experiments, discussed further below, are sensitive only to f_+ because the small electron mass makes the f_- term negligible.

76.1. $K_{\mu 3}$ Experiments

Analyses of $K_{\mu 3}$ data frequently assume a linear dependence of f_+ and f_- on t, *i.e.*,

$$f_\pm(t) = f_\pm(0)\left[1 + \lambda_\pm (t/m^2_{\pi^+})\right] \; . \tag{76.2}$$

Most $K_{\mu 3}$ data are adequately described by Eq. (76.2) for f_+ and a constant f_- (*i.e.*, $\lambda_- = 0$).

76.1.1. *Two commonly used equivalent parametrizations* :

76.1.1.1. $\lambda_+, \xi(0)$ *parametrization:*

Older analyses of $K_{\mu 3}$ data often introduce the ratio of the two form factors

$$\xi(t) = f_-(t)/f_+(t) \; . \tag{76.3}$$

The $K_{\mu 3}$ decay distribution is then described by the two parameters λ_+ and $\xi(0)$ (assuming time reversal invariance and $\lambda_- = 0$).

76.1.1.2. λ_+, λ_0 *parametrization:*

More recent $K_{\mu 3}$ analyses have parametrized in terms of the form factors f_+ and f_0, which are associated with vector and scalar exchange, respectively, to the lepton pair. f_0 is related to f_+ and f_- by

$$f_0(t) = f_+(t) + \left[t/(m^2_K - m^2_\pi)\right] f_-(t) \; . \tag{76.4}$$

Here $f_0(0)$ must equal $f_+(0)$ unless $f_-(t)$ diverges at $t = 0$. The earlier assumption that f_+ is linear in t and f_- is constant leads to f_0 linear in t:

$$f_0(t) = f_0(0)\left[1 + \lambda_0 (t/m^2_{\pi^+})\right] \; . \tag{76.5}$$

With the assumption that $f_0(0) = f_+(0)$, the two parametrizations, $(\lambda_+, \xi(0))$ and (λ_+, λ_0) are equivalent as long as correlation information is retained. (λ_+, λ_0) correlations tend to be less strong than $(\lambda_+, \xi(0))$ correlations.

Since the 2006 edition of the *Review* [4], we no longer quote results in the $(\lambda_+, \xi(0))$ parametrization. We have removed many older low statistics results from the Listings. See the 2004 version of this note [5] for these older results, and the 1982 version [6] for additional discussion of the $K^0_{\mu 3}$ parameters, correlations, and conversion between parametrizations.

76.1.2. *Quadratic Parametrization* :

More recent high-statistics experiments have included a quadratic term in the expansion of $f_+(t)$,

$$f_+(t) = f_+(0)\left[1 + \lambda'_+(t/m^2_{\pi^+}) + \frac{\lambda''_+}{2}(t/m^2_{\pi^+})^2\right] . \tag{76.6}$$

If there is a non-vanishing quadratic term, then λ_+ of Eq. (76.2) represents the average slope, which is then different from λ'_+. Our convention is to include the factor $\frac{1}{2}$ in the quadratic term, and to use m_{π^+} even for K^+_{e3} and $K^+_{\mu 3}$ decays. We have converted other's parametrizations to match our conventions, as noted in the beginning of the "$K^{\pm}_{\ell 3}$ and $K^0_{\ell 3}$ Form Factors" sections of the Listings.

76.1.3. *Pole Parametrization* :

The pole model describes the t-dependence of $f_+(t)$ and $f_0(t)$ in terms of the exchange of the lightest vector and scalar K^* mesons with masses M_v and M_s, respectively:

$$f_+(t) = f_+(0)\left[\frac{M^2_v}{M^2_v - t}\right] , \qquad f_0(t) = f_0(0)\left[\frac{M^2_s}{M^2_s - t}\right] . \tag{76.7}$$

76.1.4. *Dispersive Parametrization* :

This approach [7,8] uses dispersive techniques and the known low-energy K-π phases to parametrize the vector and scalar form factors:

$$f_+(t) = f_+(0)\mathrm{exp}\left[\frac{\mathrm{t}}{\mathrm{m}^2_\pi}(\Lambda_+ + \mathrm{H(t)})\right] ; \tag{76.8}$$

$$f_0(t) = f_+(0)\mathrm{exp}\left[\frac{\mathrm{t}}{(\mathrm{m}^2_\mathrm{K} - \mathrm{m}^2_\pi)}\,(\ln[\mathrm{C}] - \mathrm{G(t)})\right] , \tag{76.9}$$

where Λ_+ is the slope of the vector form factor, and $\ln[\mathrm{C}]=$ $\ln[\mathrm{f}_0(\mathrm{m}^2_\mathrm{K} - \mathrm{m}^2_\pi)]$ is the logarithm of the scalar form factor at the Callan-Treiman point. The functions H(t) and G(t) are dispersive integrals.

76.2. K_{e3} Experiments

Analysis of K_{e3} data is simpler than that of $K_{\mu 3}$ because the second term of the matrix element assuming a pure vector current [Eq. (76.1) above] can be neglected. Here f_+ can be assumed to be linear in t, in which case the linear coefficient λ_+ of Eq. (76.2) is determined, or quadratic, in which case the linear coefficient $\lambda_+^{'}$ and quadratic coefficient $\lambda_+^{''}$ of Eq. (76.6) are determined.

If we remove the assumption of a pure vector current, then the matrix element for the decay, in addition to the terms in Eq. (76.1), would contain

$$
\begin{aligned}
&+2m_K\, f_S\, \overline{\ell}(1+\gamma_5)\nu \\
&\qquad +(2f_T/m_K)(P_K)_\lambda (P_\pi)_\mu\, \overline{\ell}\, \sigma_{\lambda\mu}(1+\gamma_5)\nu \;, \qquad (76.10)
\end{aligned}
$$

where f_S is the scalar form factor, and f_T is the tensor form factor. In the case of the K_{e3} decays where the f_- term can be neglected, experiments have yielded limits on $|f_S/f_+|$ and $|f_T/f_+|$.

76.2.1. *Fits for $K_{\ell 3}$ Form Factors* :

For K_{e3} data, we determine best values for the three parametrizations: linear (λ_+), quadratic ($\lambda_+^{'}$, $\lambda_+^{''}$) and pole (M_v). For $K_{\mu 3}$ data, we determine best values for the three parametrizations: linear (λ_+, λ_0), quadratic ($\lambda_+^{'}, \lambda_+^{''}$, λ_0) and pole (M_v, M_s). We then assume $\mu - e$ universality so that we can combine K_{e3} and $K_{\mu 3}$ data, and again determine best values for the three parametrizations: linear (λ_+, λ_0), quadratic ($\lambda_+^{'}, \lambda_+^{''}$, λ_0), and pole (M_v, M_s). When there is more than one parameter, fits are done including input correlations. Simple averages suffice in the two K_{e3} cases where there is only one parameter: linear (λ_+) and pole (M_v).

Both KTeV and KLOE see an improvement in the quality of their fits relative to linear fits when a quadratic term is introduced, as well as when the pole parametrization is used. The quadratic parametrization has the disadvantage that the quadratic parameter $\lambda_+^{''}$ is highly correlated with the linear parameter $\lambda_+^{'}$, in the neighborhood of 95%, and that neither parameter is very well determined. The pole fit has the same number of parameters as the linear fit, but yields slightly better fit probabilities, so that it would be advisable for all experiments to include the pole parametrization as one of their choices [9].

The "Kaon Particle Listings" show the results with and without assuming μ-e universality. The "Meson Summary Tables" show all of the results assuming μ-e universality, but most results not assuming μ-e universality are given only in the Listings.

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