81. Review of Multibody Charm Analyses

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81.1. Kinematics & Models

The differential decay rate to a point $\mathbf{s} = (s_1, \ldots, s_n)$ in n dimensional phase space can be expressed as

$$d\Gamma = |\mathcal{M}(\mathbf{s})|^2 \left| \frac{\partial^n \phi}{\partial(s_1 \ldots s_n)} \right| d^n s \tag{81.1}$$

where $|\partial^n \phi / \partial(s_1 \ldots s_n)|$ represents the density of states at $\mathbf{s}$, and $\mathcal{M}$ the matrix element for the decay at that point in phase space, which is $2, 5, 8, \ldots$ dimensional for D decays to $3, 4, 5, \ldots$ spinless particles. Additional parameters are required to fully describe decays involving particles with non-zero spin in the initial or final state.

For the important case of D decays to 3 pseudoscalars, the decay kinematics can be represented in a two dimensional Dalitz plot [1]. This is usually parametrized in terms of $s_{12} \equiv (p_1 + p_2)^2$ and $s_{23} \equiv (p_2 + p_3)^2$, where p_1, p_2, p_3 are the four-momenta of the final state particles. In terms of these variables, phase-space density is constant across the kinematically allowed region, so that any structure seen in the Dalitz plot is a direct consequence of the dynamics encoded in $|\mathcal{M}|^2$. Note that here, because the 3-momenta of the decay products are confined to a plane, no parity violating kinematic observables can be constructed (unless they also violate rotational invariance). This is not the case for decays to four or more particles. These can therefore not be unambiguously described in terms of analogously-defined variables s_{ij}, s_{ijk}, which are parity-even. The use of parity-odd observables in four body decays is discussed below.

In the widely-used isobar approach, the matrix element $\mathcal{M}$ is modeled as a sum of interfering decay amplitudes, each proceeding through resonant two-body decays [2]. See Refs. 2–4 for a review of resonance phenomenology. In most analyses, each resonance is described by a Breit-Wigner [5] or Flatté [6] lineshape, and the model includes a non-resonant term with a constant phase and magnitude. This approach has well-known theoretical limitations, such as the violation of unitarity and analyticity, which can break the relationship between magnitude and phase across phase space. This motivates the use of more sophisticated descriptions, especially for broad, overlapping resonances (frequently found in S-wave components) where these limitations are particularly problematic. In charm analyses, these approaches have included the K-matrix approach [5–8] which respects two-body unitarity; the use of LASS scattering data [9]; dispersive methods [10–13]; methods based on chiral symmetry [14–16], QCD factorisation (although this seems better suited to B decays) [17–19]; and quasi model-independent parametrizations which use generic lineshapes, with minimal theory input and many free parameters, for a subset of resonances [20–23]. An important example, with a rich resonance structure, is $D^0 \to K_S \pi^+ \pi^-$, which is a key channel in Charge-Parity (CP) violation and charm mixing analyses. The first analysis by CLEO [24] described the Dalitz plot with 5k signal events with 10 resonant components. This and later analyses by Belle [25] and CDF [26] model the Dalitz plot as a sum of Breit Wigner and Flatté line shapes, and a non-resonant component. BaBar [27] on the other hand use a K-matrix description for the $\pi\pi$ S-wave based on [28] and input from LASS scattering data for the $K-\pi$ S-wave, with no need to add a non-resonant component to describe the data. This approach is also followed in the latest analysis of this channel, published jointly by BaBar and Belle [29]. In total 18 resonant components, including four doubly Cabibbo suppressed ones, are required to describe the Dalitz plot with 1.1M $D^0 \to K_S\pi^+\pi^-$ events. Belle's and BaBar's data have been re-analyzed by [17] in a QCD factorization framework, using line-shape parametrizations for the S [30,31] and P wave [11] contributions that preserve 2-body unitarity and analyticity. The measurements give compatible results for the components they share.

The field of amplitude analyses remains very active. Publications since the last update of this review two years ago include Dalitz plot analyses of $D_s^+ \to \pi^+\pi^0\eta$ by BES III [32]; $D^+ \to K^+K^-K^+$ by LHCb [33]; $D^0 \to \pi^+\pi^-\pi^0$ by BaBar [34]; BaBar and Belle's joint analysis of $D^0 \to K_S\pi^+\pi^-$ [29]; and a re-analysis of $D^+ \to K_S\pi^0\pi^+$ and $D^+ \to K^-\pi^+\pi^+$ data from FOCUS, CLEO and BES III by Niecknig and Kubis [13]. Ahn, Yang and Nam developed amplitude models for $\Lambda_c^+ \to K^-p\pi^+$ and $\Lambda_c^+ \to K_Sp\pi^0$ [35] based on BELLE data [36]. There has also been significant progress in four body amplitude analyses: $D^0 \to K^+K^-\pi^+\pi^-$ and $D^0 \to \pi^+\pi^-\pi^+\pi^-$ using CLEO data [22]; $D^+ \to K_S\pi^+\pi^+\pi^-$, $D^0 \to K^-\pi^+\pi^-\pi^+$ and $D^0 \to K^-\pi^+\pi^0\pi^0$ by BES III [37–39]; and $D^0 \to K^+K^-\pi^+\pi^-$, $D^0 \to K^\mp\pi^\pm\pi^\mp\pi^\pm$ [23,40] by LHCb. Noteworthy is the increasing sophistication of recent amplitude analyses, most of which go substantially beyond the isobar model with Breit Wigner and Flatté lineshapes. However, with the notable exception of [13] and [33], they remain within the isobar framework which describes the decay as a series of 2-body processes; even if these are modeled with increasing sophistication, the approach ignores long-range hadronic effects such as re-scattering and does not respect 3 (or 4)-body unitarity and analyticity.

Several groups work on improved models. Dispersive techniques, which respect 3-body unitarity and analyticity, have been successfully applied to regions of the $D^+ \to K^-\pi^+\pi^+$ and $D^+ \to K_S\pi^0\pi^+$ Dalitz plots below the $\eta'K$ threshold [12,13], where they provide a good description of the data with fewer fit parameters than the isobar approach. Ref. [41] uses a unitary coupled channel approach to describe $D^+ \to K^-\pi^+\pi^+$, which has no restrictions on the kinematic range, but requires additional parameters to describe the Dalitz plot above the $\eta'K$ threshold. Using an effective chiral Lagrangian, the authors of Ref. [16] provide a description of the annihilation contribution to the decay amplitude which respects 3-body unitarity. This approach provides a good description of LHCb $D^+ \to K^+K^-K^+$ data, with fewer parameters than an equivalent isobar model [33].

Limitations in the theoretical description of interfering resonances are the leading source of systematic uncertainty in many analyses. This is set to become increasingly problematic given the statistical precision achievable with the vast, clean charm samples available at the B factories, LHCb, and their upgrades. In some cases, the model uncertainty can be removed through model-independent methods, often relying on input from the charm threshold, as discussed below. Ref. 42 expand the scope and applicability of the quasi model-independent approach in amplitude fits. At the same time, increasingly sophisticated models are being developed, and applied to data.

81.2. Applications of multibody charm analyses

Amplitude analyses provide sensitivity to both relative magnitudes and phases of the interfering decay amplitudes. It is especially this sensitivity to phases that makes amplitude analyses such a uniquely powerful tool for studying a wide range of phenomena. Here we concentrate on their use for *CP* violation and mixing measurements in charm, and charm inputs to *CP* violation analyses in B meson decays (see also [43,44]). The properties of light-meson resonances determined in D amplitude analyses are reported in the light-unflavored-meson section of this *Review*.

81.2.1. *Time-integrated searches for CP violation in charm* :

Comparing the results of amplitude fits for *CP*-conjugate decay modes provides a measure of *CP* violation. Recent *CP* violation searches using this method include amplitude analyses of $D^0 \to K^0_S K^\pm \pi^\mp$ and $D^0 \to K^+K^-\pi^+\pi^-$ by LHCb [45,40], and $D^0 \to K^+K^-\pi^+\pi^-$, $D^0 \to \pi^+\pi^-\pi^+\pi^-$ [46,22] using CLEO data.

A widely-used amplitude model-independent technique to search for local *CP* violation is based on performing a χ^2 comparison of *CP*-conjugate phase-space distributions. This method was pioneered by BaBar [47] and developed further in [48–50], with recent results reported by BaBar [51] and LHCb in $D^\pm \to K^+K^-\pi^\pm$ [52,53], CDF in $D^0 \to K_S\pi^+\pi^-$ [26], and LHCb in $D^+ \to \pi^-\pi^+\pi^+$ [55], $D^0 \to K^+K^-\pi^+\pi^-$ and $D^0 \to \pi^+\pi^-\pi^+\pi^-$ [50]. Un-binned methods can increase the sensitivity [54] and have been applied by LHCb to $D^+ \to \pi^-\pi^+\pi^+$, $D^0 \to \pi^+\pi^-\pi^0$ and $D^0 \to \pi^+\pi^-\pi^+\pi$ [55,56,73].

An alternative model-independent approach is based on constructing observables in four body decays that are odd under motion reversal ("naïve T") [58–66], which is equivalent to P for scalar particles [66]. One such observable is $C_T = \vec{p}_2 \cdot (\vec{p}_3 \times \vec{p}_4) = (1/m_D)\epsilon_{\alpha\beta\gamma\delta}\, p_1^\alpha p_2^\beta p_3^\gamma p_4^\delta$, where $\vec{p}_i$ are the decay products' three momenta in the decay's restframe, and p_i are their four-momenta. Identical particles (as in $D^0 \to K^+\underline{\pi^-}\pi^+\underline{\pi^-}$) are ordered by momentum magnitude. Comparing the P violating asymmetry $A_T \equiv \frac{\Gamma(C_T>0)-\Gamma(C_T<0)}{\Gamma(C_T>0)+\Gamma(C_T<0)}$ with its C-conjugate in $\overline{D}^0$ decays, provides sensitivity to *CP* violation. Searches for *CP* violation in this manner have been carried out for $D^0 \to K^+K^-\pi^+\pi^-$ by FOCUS, BaBar, LHCb and Belle [67,68,69,70], where LHCb increase the sensitivity of the method by analysing the data in bins of phase space, and Belle's analysis considers several new, hitherto unused P-odd variables; $D^+ \to K^+K_S\pi^+\pi^-$ and $D_s^+ \to K^+K_S\pi^+\pi^-$by BaBar [71]; and $D^0 \to K_S\pi^+\pi^-\pi^0$ by Belle [72]. LHCb's unbinned comparison of kinematic distributions in $D^0, \overline{D}^0 \to \pi^+\pi^-\pi^+\pi^-$ is sensitive to *CP* violation in both P even and P-odd kinematic variables [73].

The results of all measurements described in this section are compatible with *CP* conservation in charm. Given the recent discovery of *CP* violation in $D^0 \to K^+K^-$, $D^0 \to \pi^+\pi^-$ decays, and in view of the vast data samples about to be collected, one might expect this to change in the foreseeable future.

81.2.2. *Charm Mixing and CP violation* : Time-dependent amplitude analyses in decays to final states that are accessible to both D^0 and $\overline{D}^0$ have unique sensitivity to mixing parameters. A Dalitz plot analysis of a self-conjugate final state, such as $K_S\pi^+\pi^-$ and $K_SK^+K^-$, allows the measurement of the phase difference between the relevant D^0 and $\overline{D}^0$ decay amplitudes, and thus a direct measurement of x and y, the normalised mass and width difference of the $D^0 - \overline{D}^0$ system's mass eigenstates. This is in contrast to decays like $D^0 \to K\pi$ [74] which only provide access to the decay-specific parameters x'^2, y'. Multibody charm analyses are also sensitive to CP violation in mixing and in the interference between mixing and decay; these results are summarised in [43,44].

81.2.3. *CP violation in decays of Beauty to Charm* : Neutral D mesons originating from $B^- \to DK^-$ (here denoted as D_{B^-}) are a superposition of D^0 and $\overline{D}^0$ with a relative phase that depends on the CKM unitarity triangle parameter γ/ϕ_3,

$$D_{B^-} \propto D^0 + r_B e^{i(\delta_B - \gamma)}\overline{D}^0,$$

where δ_B is a CP conserving strong phase, and $r_B \sim 0.1$. In the corresponding CP-conjugate expression, γ/ϕ_3 changes sign. An amplitude analysis of the subsequent decay of the $D_{B^\pm}$ to a state accessible to both D^0 and $\overline{D}^0$ allows the measurement of γ/ϕ_3 [75–79]. The method generalizes to similar B hadron decays, such as $B^0 \to DK^{*0}$. Measurements based on this technique have been reported by BaBar [80,81], Belle [25,82] and LHCb [83–92]. The most precise individual results come from the study of $D_{B^-} \to K_S\pi^+\pi^-$ and $D_{B^-} \to K_SK^+K^-$ with an uncertainty of $\sim 10°$ [25,80,82,86,92]; combining measurements in multiple decay modes leads to a current uncertainty on γ/ϕ_3 of less than $6°$.

The interference between mixing and decay in $B^0 \to D^0h^0$ with $h^0 = \pi^0, \eta, \omega$ provides sensitivity to β, which can be extracted from the Dalitz plot of the subsequent $D^0 \to K_S\pi^+\pi^-$ decay [29,93–96]. The combined BaBar/Belle analysis based on this technique resolved the ambiguity in β present in other measurements, such as $B^0 \to J/\psi K_S$, in favour of the solution compatible with other unitarity triangle constraints [29].

81.3. Model Independent Methods and the Charm Threshold

The precision measurement of mixing or CP violation parameters such γ/ϕ_3 from multibody charm decays requires as input the phase-differences between the D^0 and $\overline{D}^0$ amplitudes across phase space, as well as their magnitudes, for each final state of interest. While the magnitudes are fairly easily measured, the phase information requires either amplitude models with reliable phase motion, or model-independent approaches.

Model-independent measurements of the relevant phase differences rely on interference effects in the decays of well-defined coherent superpositions of D^0 and $\overline{D}^0$. These are accessible at the charm threshold, where CLEO–c and BES III operate [43,97–104]. Charm mixing also results in a (time-dependent) $D^0 - \overline{D}^0$ superposition, that can be used to measure the relevant phase information as input to γ/ϕ_3 measurements. This method is particularly powerful in doubly Cabibbo-suppressed decays such as $D^0 \to K^+\pi^-\pi^+\pi^-$, and when used in combination with threshold data [105,106]. Under some circumstances, with large data sets, the relevant strong phases and γ/ϕ_3 can be extracted simultaneously without external input, for example in simultaneous analysis of the $B^0 \to DK^+\pi^-$ Dalitz plot and that of the subsequent $D \to K^0_{\rm S}\pi^+\pi^-$ decay [115]. However, the global effort to achieve a measurement of γ/ϕ_3 to sub-degree precision will continue to rely critically on input from the charm threshold.

The model-independent phase information is provided either integrated over the entire phase space of the decay, or in sub-regions/bins. The results can be expressed in terms of one complex parameter $\mathcal{Z} = Re^{-i\delta} = c + is$ per pair of CP-conjugate bins, with magnitude $R \leq 1$. Larger R values lead to higher sensitivity to γ/ϕ_3. Amplitude models can be used to optimise the binning for sensitivity to γ/ϕ_3, without introducing a model-dependent bias in the result.

CLEO–c data have been analyzed to provide binned $\mathcal{Z}$ for the self-conjugate decays $D^0 \to K_S\pi^+\pi^-$, $D^0 \to K_S K^+K^-$, $D^0 \to \pi^+\pi^-\pi^+\pi^-$, and $D^0 \to K_S\pi^-\pi^+\pi^0$ [107–110]; and phase space-integrated values for $D^0, \overline{D}{}^0 \to K_S K^+\pi^-$, $K^+\pi^-\pi^0$ and $K^+\pi^-\pi^+\pi^-$ [111,112]. Adding input from LHCb's charm mixing analysis significantly improves the constraints on $\mathcal{Z}$ for $D^0, \overline{D}{}^0 \to K^+\pi^-\pi^+\pi^-$ [112,113]. A recent study based on LHCb's $D^0, \overline{D}{}^0 \to K^+\pi^-\pi^+\pi^-$ amplitude models [23] and CLEO–c data indicates that a binned analysis of $D^0, \overline{D}{}^0 \to K^+\pi^-\pi^+\pi^-$ could lead to the most precise individual measurement of γ/ϕ_3 [114]. For self-conjugate decays such as $D^0 \to \pi^+\pi^-\pi^0$, analysed with a single pair of bins, $\mathcal{Z}$ is real-valued, and usually expressed in terms of the CP-even fraction $F_+ \equiv \frac{1}{2}\left(\mathrm{Re}(\mathcal{Z}) + 1\right)$, defined such that a CP-even eigenstate has $F_+ = 1$, while a CP-odd eigenstate has $F_+ = 0$ [102]. Recent analyses of CLEO–c data reveal that $D^0 \to \pi^+\pi^-\pi^0$ is compatible with being completely CP-even with $F_+ = 0.973 \pm 0.017$, while $D^0 \to K^+K^-\pi^0$ has $F_+ = 0.732 \pm 0.055$, $D^0 \to \pi^+\pi^-\pi^+\pi^-$ has $F_+ = 0.769 \pm 0.023$ and $D^0 \to K_S\pi^+\pi^-\pi^0$ has 0.238 ± 0.020 [103,109,110].

It is interesting to compare these values with those obtained from amplitude models as a cross check of the models' phase-motion. $F_+^{4\pi\ \rm model} = 0.729 \pm 0.020$ calculated from Ref. 22's $D^0 \to \pi^+\pi^-\pi^+\pi^-$ model, compares well to the measured value given above, as does $\mathcal{Z}^{K3\pi\ \rm model} = 0.459 \pm 0.025$ [23] to $\mathcal{Z}^{K3\pi\ \rm meas} = (0.32^{+0.17}_{-0.13})\exp(-i(128^\circ{}^{+28^\circ}_{-17^\circ}))$ [112,113]. Binwise comparisons for $D^0 \to K_S\pi^+\pi^-$, $D^0 \to K_S K^+K^-$, $D^0 \to \pi^+\pi^-\pi^+\pi^-$, and $D^0, \overline{D}{}^0 \to K^+\pi^-\pi^+\pi^-$ can be found in [107–109,114].

81.4. Summary

Multibody charm decays offer a rich phenomenology, including unique sensitivity to CP violation and charm mixing. This is a highly dynamic field with many new results (some of which we presented here) and rapidly increasing, high quality datasets. These datasets constitute a huge opportunity, but also a challenge to improve the theoretical descriptions of soft hadronic effects in multibody decays. For some measurements, model-independent methods, many relying on input from the charm threshold, provide a way of removing model-induced uncertainties. At the same time, substantial progress in the theoretical description of multibody decays is being made.

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