7. Electromagnetic Relations

Revised September 2005 by H.G. Spieler (LBNL).

Quantity	Gaussian CGS	SI
Conversion factors: Charge: Potential: Magnetic field:	 $2.997\,924\,58 \times 10^9$ esu $(1/299.792\,458)$ statvolt (ergs/esu) 10^4 gauss $= 10^4$ dyne/esu	 $= 1$ C $= 1$ A s $= 1$ V $= 1$ J C^{-1} $= 1$ T $= 1$ N A^{-1}m^{-1}
	$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)$	$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$
	$\boldsymbol{\nabla} \cdot \mathbf{D} = 4\pi\rho$ $\boldsymbol{\nabla} \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \frac{4\pi}{c}\mathbf{J}$ $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ $\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0$	$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho$ $\boldsymbol{\nabla} \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}$ $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ $\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$
Constitutive relations:	$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \quad \mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$	$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}, \quad \mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$
Linear media:	$\mathbf{D} = \epsilon\mathbf{E}, \quad \mathbf{H} = \mathbf{B}/\mu$ 1 1	$\mathbf{D} = \epsilon\mathbf{E}, \quad \mathbf{H} = \mathbf{B}/\mu$ $\epsilon_0 = 8.854\,187\ldots \times 10^{-12}$ F m^{-1} $\mu_0 = 4\pi \times 10^{-7}$ N A^{-2}
	$\mathbf{E} = -\boldsymbol{\nabla}V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$ $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$	$\mathbf{E} = -\boldsymbol{\nabla}V - \frac{\partial \mathbf{A}}{\partial t}$ $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$
	$V = \sum_{\text{charges}} \frac{q_i}{r_i} = \int \frac{\rho(\mathbf{r}')}{\lvert\mathbf{r} - \mathbf{r}'\rvert}\, d^3x'$ $\mathbf{A} = \frac{1}{c}\oint \frac{I\, d\boldsymbol{\ell}}{\lvert\mathbf{r} - \mathbf{r}'\rvert} = \frac{1}{c}\int \frac{\mathbf{J}(\mathbf{r}')}{\lvert\mathbf{r} - \mathbf{r}'\rvert}\, d^3x'$	$V = \frac{1}{4\pi\epsilon_0} \sum_{\text{charges}} \frac{q_i}{r_i} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\lvert\mathbf{r} - \mathbf{r}'\rvert}\, d^3x'$ $\mathbf{A} = \frac{\mu_0}{4\pi}\oint \frac{I\, d\boldsymbol{\ell}}{\lvert\mathbf{r} - \mathbf{r}'\rvert} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\lvert\mathbf{r} - \mathbf{r}'\rvert}\, d^3x'$
	$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}$ $\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \frac{1}{c}\mathbf{v} \times \mathbf{B})$ $\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}$ $\mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \frac{1}{c}\mathbf{v} \times \mathbf{E})$	$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}$ $\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B})$ $\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}$ $\mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E})$

$$\frac{1}{4\pi\epsilon_0} = c^2 \times 10^{-7}\ \text{N A}^{-2} = 8.987\,55\ldots \times 10^9\ \text{m F}^{-1}\ ;\ \frac{\mu_0}{4\pi} = 10^{-7}\ \text{N A}^{-2}\ ;\ c = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 2.997\,924\ 58 \times 10^8\ \text{m s}^{-1}$$

7.1. Impedances (SI units)

ρ = resistivity at room temperature in $10^{-8}\,\Omega$ m:

~ 1.7 for Cu	~ 5.5 for W
~ 2.4 for Au	~ 73 for SS 304
~ 2.8 for Al	~ 100 for Nichrome

(Al alloys may have double the Al value.)

For alternating currents, instantaneous current I, voltage V, angular frequency ω:

$$V = V_0\, e^{j\omega t} = ZI\ . \tag{7.1}$$

Impedance of self-inductance L: $Z = j\omega L$.

Impedance of capacitance C: $Z = 1/j\omega C$.

Impedance of free space: $Z = \sqrt{\mu_0/\epsilon_0} = 376.7\ \Omega$.

High-frequency surface impedance of a good conductor:

$$Z = \frac{(1+j)\,\rho}{\delta}\ , \quad \text{where } \delta = \text{skin depth}\ ; \tag{7.2}$$

$$\delta = \sqrt{\frac{\rho}{\pi\nu\mu}} \approx \frac{6.6\ \text{cm}}{\sqrt{\nu\,(\text{Hz})}} \quad \text{for Cu}\ . \tag{7.3}$$

7.2. Capacitors, inductors, and transmission Lines

The capacitance between two parallel plates of area A spaced by the distance d and enclosing a medium with the dielectric constant ε is

$$C = K\varepsilon A/d\,, \tag{7.4}$$

where the correction factor K depends on the extent of the fringing field. If the dielectric fills the capacitor volume without extending beyond the electrodes. the correction factor $K \approx 0.8$ for capacitors of typical geometry.

The inductance at high frequencies of a straight wire whose length ℓ is much greater than the wire diameter d is

$$L \approx 2.0 \left[\frac{\text{nH}}{\text{cm}}\right] \cdot \ell \left(\ln\left(\frac{4\ell}{d}\right) - 1\right)\ . \tag{7.5}$$

For very short wires, representative of vias in a printed circuit board, the inductance is

$$L(\text{in nH}) \approx \ell/d\,. \tag{7.6}$$

A transmission line is a pair of conductors with inductance L and capacitance C. The characteristic impedance $Z = \sqrt{L/C}$ and the phase velocity $v_p = 1/\sqrt{LC} = 1/\sqrt{\mu\varepsilon}$, which decreases with the inverse square root of the dielectric constant of the medium. Typical coaxial and ribbon cables have a propagation delay of about 5 ns/cm.

The impedance of a coaxial cable with outer diameter D and inner diameter d is

$$Z = 60\,\Omega \cdot \frac{1}{\sqrt{\varepsilon_r}} \ln\frac{D}{d}\,, \tag{7.7}$$

where the relative dielectric constant $\varepsilon_r = \varepsilon/\varepsilon_0$. A pair of parallel wires of diameter d and spacing $a > 2.5\,d$ has the impedance

$$Z = 120\,\Omega \cdot \frac{1}{\sqrt{\varepsilon_r}} \ln\frac{2a}{d}\,. \tag{7.8}$$

This yields the impedance of a wire at a spacing h above a ground plane,

$$Z = 60\,\Omega \cdot \frac{1}{\sqrt{\varepsilon_r}} \ln\frac{4h}{d}\,. \tag{7.9}$$

A common configuration utilizes a thin rectangular conductor above a ground plane with an intermediate dielectric (microstrip). Detailed calculations for this and other transmission line configurations are given by Gunston.*

* M.A.R. Gunston. Microwave Transmission Line Data, Noble Publishing Corp., Atlanta (1997) ISBN 1-884932-57-6, TK6565.T73G85.

7.3. Synchrotron radiation (CGS units)

For a particle of charge e, velocity $v = \beta c$, and energy $E = \gamma mc^2$, traveling in a circular orbit of radius R, the classical energy loss per revolution δE is

$$\delta E = \frac{4\pi}{3}\,\frac{e^2}{R}\,\beta^3\,\gamma^4\ . \tag{7.10}$$

For high-energy electrons or positrons ($\beta \approx 1$), this becomes

$$\delta E\ (\text{in MeV}) \approx 0.0885\ [E(\text{in GeV})]^4/R(\text{in m})\ . \tag{7.11}$$

For $\gamma \gg 1$, the energy radiated per revolution into the photon energy interval $d(\hbar\omega)$ is

$$dI = \frac{8\pi}{9}\alpha\,\gamma\,F(\omega/\omega_c)\,d(\hbar\omega)\ , \tag{7.12}$$

where $\alpha = e^2/\hbar c$ is the fine-structure constant and

$$\omega_c = \frac{3\gamma^3 c}{2R} \tag{7.13}$$

is the critical frequency. The normalized function $F(y)$ is

$$F(y) = \frac{9}{8\pi}\sqrt{3}\,y \int_y^\infty K_{5/3}\,(x)\ dx\ , \tag{7.14}$$

where $K_{5/3}\,(x)$ is a modified Bessel function of the third kind. For electrons or positrons,

$$\hbar\omega_c\,(\text{in keV}) \approx 2.22\ [E(\text{in GeV})]^3/R(\text{in m})\ . \tag{7.15}$$

Fig. 7.1 shows $F(y)$ over the important range of y.

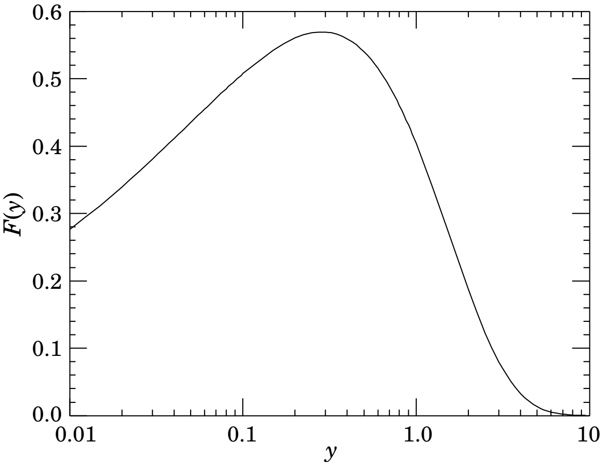

Figure 7.1: The normalized synchrotron radiation spectrum $F(y)$.

For $\gamma \gg 1$ and $\omega \ll \omega_c$,

$$\frac{dI}{d(\hbar\omega)} \approx 3.3\alpha\,(\omega R/c)^{1/3}\ , \tag{7.16}$$

whereas for

$$\gamma \gg 1 \text{ and } \omega \gtrsim 3\omega_c\ ,$$

$$\frac{dI}{d(\hbar\omega)} \approx \sqrt{\frac{3\pi}{2}}\,\alpha\,\gamma\left(\frac{\omega}{\omega_c}\right)^{1/2} e^{-\omega/\omega_c}\left[1 + \frac{55}{72}\frac{\omega_c}{\omega} + \ldots\right]\ . \tag{7.17}$$

The radiation is confined to angles $\lesssim 1/\gamma$ relative to the instantaneous direction of motion. For $\gamma \gg 1$, where Eq. (7.12) applies, the mean number of photons emitted per revolution is

$$N_\gamma = \frac{5\pi}{\sqrt{3}}\alpha\gamma\ , \tag{7.18}$$

and the mean energy per photon is

$$\langle\hbar\omega\rangle = \frac{8}{15\sqrt{3}}\hbar\omega_c\ . \tag{7.19}$$

When $\langle\hbar\omega\rangle \gtrsim \mathrm{O}(E)$, quantum corrections are important.

See J.D. Jackson, *Classical Electrodynamics*, 3rd edition (John Wiley & Sons, New York, 1998) for more formulae and details. (Note that earlier editions had ω_c twice as large as Eq. (7.13).