7. Electromagnetic Relations

Revised September 2005 by H.G. Spieler (LBNL).

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Gaussian CGS</th>
<th>SI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conversion factors:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Charge:</td>
<td>2.997 921 58 \times 10^9 \text{ esu}</td>
<td>= 1 \text{ C} = 1 \text{ A s}</td>
</tr>
<tr>
<td>Potential:</td>
<td>(1/299.792 458) \text{ statvolt (ergs/esu)}</td>
<td>= 1 \text{ V} = 1 \text{ J C}^{-1}</td>
</tr>
<tr>
<td>Magnetic field:</td>
<td>10^4 \text{ gauss} = 10^4 \text{ dyne/esu}</td>
<td>= 1 \text{ T} = 1 \text{ N A}^{-1} \text{ m}^{-1}</td>
</tr>
<tr>
<td></td>
<td>F = q (E + \nabla \times B)</td>
<td>F = q (E + v \times B)</td>
</tr>
<tr>
<td></td>
<td>\nabla \cdot D = 4\pi \rho</td>
<td>\nabla \cdot D = \rho</td>
</tr>
<tr>
<td></td>
<td>\nabla \times H - \frac{1}{c} \frac{\partial D}{\partial t} = 4\pi J</td>
<td>\nabla \times H - \frac{\partial D}{\partial t} = J</td>
</tr>
<tr>
<td></td>
<td>\nabla \cdot B = 0</td>
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<tr>
<td></td>
<td>\nabla \times E + \frac{1}{c} \frac{\partial B}{\partial t} = 0</td>
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</tr>
<tr>
<td>Constitutive relations:</td>
<td>D = \epsilon_0 E + P, H = B - \frac{4\pi}{c} M</td>
<td>D = \epsilon_0 E + P, H = B/\mu_0 - M</td>
</tr>
<tr>
<td>Linear media:</td>
<td>D = \epsilon_0 E, H = B/\mu</td>
<td>\epsilon_0 = 8.854 187 \ldots \times 10^{-12} \text{ F m}^{-1}</td>
</tr>
<tr>
<td></td>
<td>1 \phantom{xdx}</td>
<td>\mu_0 = 4\pi \times 10^{-7} \text{ N A}^{-2}</td>
</tr>
<tr>
<td></td>
<td>1 \phantom{xdx}</td>
<td></td>
</tr>
<tr>
<td></td>
<td>E = -\nabla V - \frac{1}{c} \frac{\partial A}{\partial t}</td>
<td>E = -\nabla V - \frac{\partial A}{\partial t}</td>
</tr>
<tr>
<td></td>
<td>B = \nabla \times A</td>
<td>B = \nabla \times A</td>
</tr>
<tr>
<td></td>
<td>V = \sum_{\text{charges}} \frac{q_i}{\epsilon_0} \int \frac{\rho (r')}{</td>
<td>r - r'</td>
</tr>
<tr>
<td></td>
<td>A = \frac{1}{c} \int \frac{\mathbf{l} \cdot d\ell}{</td>
<td>r - r'</td>
</tr>
<tr>
<td></td>
<td>E'_{</td>
<td></td>
</tr>
<tr>
<td></td>
<td>E'<em>{\perp} = \gamma (E</em>{\perp} + \frac{1}{c} v \times B)</td>
<td>E'<em>{\perp} = \gamma (E</em>{\perp} + v \times B)</td>
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<tr>
<td></td>
<td>B'_{</td>
<td></td>
</tr>
<tr>
<td></td>
<td>B'<em>{\perp} = \gamma (B</em>{\perp} - \frac{1}{c} v \times E)</td>
<td>B'<em>{\perp} = \gamma (B</em>{\perp} - v \times E)</td>
</tr>
</tbody>
</table>

\[
\frac{1}{4\pi \epsilon_0} = \epsilon^2 \times 10^{-7} \text{ N A}^{-2} = 8.987 55 \ldots \times 10^9 \text{ m F}^{-1}; \quad \frac{\mu_0}{4\pi} = 10^{-7} \text{ N A}^{-2}; \quad \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 2.997 924 \ 58 \times 10^8 \text{ m s}^{-1}
\]
7.1. Impedances (SI units)

\( \rho = \text{resistivity at room temperature in } 10^{-8} \Omega \text{ m} \):

\begin{align*}
\sim 1.7 & \text{ for Cu} \quad \sim 5.5 \text{ for W} \\
\sim 2.4 & \text{ for Au} \quad \sim 73 \text{ for SS 304} \\
\sim 2.8 & \text{ for Al} \quad \sim 100 \text{ for Nichrome} \\
\end{align*}

(AI alloys may have double the Al value.)

For alternating currents, instantaneous current \( I \), voltage \( V \), angular frequency \( \omega \):

\[ V = V_0 \, e^{j\omega t} = ZI \, . \tag{7.1} \]

Impedance of self-inductance \( L \): \( Z = j\omega L \).

Impedance of capacitance \( C \): \( Z = 1/j\omega C \).

High-frequency surface impedance of a good conductor:

\[ Z = \left( \frac{1 + j}{j} \right) \rho / \delta, \quad \text{where } \delta = \text{skin depth} \, ; \tag{7.2} \]

\[ \delta = \sqrt{\frac{\rho}{\pi v_0}} \approx \frac{6.6 \text{ cm}}{\sqrt{\nu (\text{Hz})}} \text{ for Cu} \, . \tag{7.3} \]

7.2. Capacitors, inductors, and transmission Lines

The capacitance between two parallel plates of area \( A \) spaced by the distance \( d \) and enclosing a medium with the dielectric constant \( \varepsilon \) is

\[ C = K \varepsilon A/d \, , \tag{7.4} \]

where the correction factor \( K \) depends on the extent of the fringing field. If the dielectric fills the capacitor volume without extending beyond the electrodes, the correction factor \( K \approx 0.8 \) for capacitors of typical geometry.

The inductance at high frequencies of a straight wire whose length \( \ell \) is much greater than the wire diameter \( d \) is

\[ L \approx 2.6 \frac{\text{mH}}{\text{cm}} \cdot \ell \left( \ln \frac{4\ell}{d} - 1 \right) \, . \tag{7.5} \]

For very short wires, representative of vias in a printed circuit board, the inductance is

\[ L(\text{in nH}) \approx \ell / d \, . \tag{7.6} \]

A transmission line is a pair of conductors with inductance \( L \) and capacitance \( C \). The characteristic impedance \( Z = \sqrt{L/C} \) and the phase velocity \( v_p = 1/\sqrt{LC} = 1/\sqrt{\mu \varepsilon} \), which decreases with the inverse square root of the dielectric constant of the medium. Typical coaxial and ribbon cables have a propagation delay of about 5 ns/cm.

The impedance of a coaxial cable with outer diameter \( D \) and inner diameter \( d \) is

\[ Z = 60 \Omega \cdot \frac{1}{\sqrt{\varepsilon_r}} \ln \frac{D}{d} \, . \tag{7.7} \]

where the relative dielectric constant \( \varepsilon_r = \varepsilon / \varepsilon_0 \). A pair of parallel wires of diameter \( d \) and spacing \( a > 2.5d \) has the impedance

\[ Z = 120 \Omega \cdot \frac{1}{\sqrt{\varepsilon_r}} \ln \frac{2a}{d} \, . \tag{7.8} \]

This yields the impedance of a wire at a spacing \( h \) above a ground plane,

\[ Z = 60 \Omega \cdot \frac{1}{\sqrt{\varepsilon_r}} \ln \frac{4h}{a} \, . \tag{7.9} \]

A common configuration utilizes a thin rectangular conductor above a ground plane with an intermediate dielectric (microstrip). Detailed calculations for this and other transmission line configurations are given by Gunston.*


7.3. Synchrotron radiation (CGS units)

For a particle of charge \( e \), velocity \( v = \beta c \), and energy \( E = \gamma mc^2 \), traveling in a circular orbit of radius \( R \), the classical energy loss per revolution \( \delta E \) is

\[ \delta E = \frac{4\pi}{3} \frac{e^2}{R} \beta^3 \gamma^4 \, . \tag{7.10} \]

For high-energy electrons or positrons (\( \beta \approx 1 \)), this becomes

\[ \delta E \text{ (in MeV)} \approx 0.0885 \left[ E \text{ (in GeV)} \right]^3 / R \text{ (in m)} \, . \tag{7.11} \]

For \( \gamma \gg 1 \), the energy radiated per revolution into the photon energy interval \( d(\hbar \nu) \) is

\[ dI = \frac{8\pi}{9} \alpha \gamma F(\omega/\omega_c) \, d(\hbar \nu) \, , \tag{7.12} \]

where \( a = e^2/\hbar c \) is the fine-structure constant and

\[ \omega_c = \frac{3\gamma^3 e}{2R} \, . \tag{7.13} \]

is the critical frequency. The normalized function \( F(y) \) is

\[ F(y) = \frac{9}{8\pi} \sqrt{3} y \int_y^\infty K_{5/3} (x) \, dx \, , \tag{7.14} \]

where \( K_{5/3} (x) \) is a modified Bessel function of the third kind. For electrons or positrons, \( \hbar \omega_c \text{ (in keV)} \approx 2.22 \left[ E \text{ (in GeV)} \right]^3 / R \text{ (in m)} \, . \tag{7.15} \]

Fig. 7.1 shows \( F(y) \) over the important range of \( y \).

![Figure 7.1: The normalized synchrotron radiation spectrum \( F(y) \).](image)

For \( \gamma \gg 1 \) and \( \omega \ll \omega_c \),

\[ \frac{dI}{d(\hbar \nu)} \approx 3.3a \left( \omega R/c \right)^{1/3} \, , \tag{7.16} \]

whereas for

\[ \gamma \gg 1 \text{ and } \omega \gtrsim 3\omega_c \, , \]

\[ \frac{dI}{d(\hbar \nu)} \approx \sqrt{\frac{3\pi}{2}} \alpha \gamma \left( \frac{\omega}{\omega_c} \right)^{1/2} e^{-\omega/\omega_c} \left[ 1 + \frac{55}{72} (\omega/\omega_c) \right] \, . \tag{7.17} \]

The radiation is confined to angles \( \lesssim 1/\gamma \) relative to the instantaneous direction of motion. For \( \gamma \gg 1 \), where Eq. (7.12) applies, the mean number of photons emitted per revolution is

\[ N_\gamma = \frac{5\pi}{12} \alpha \gamma \, , \tag{7.18} \]

and the mean energy per photon is

\[ \langle \hbar \nu \rangle = \frac{8}{15 \sqrt{3}} \hbar \omega_c \, . \tag{7.19} \]

When \( \langle \hbar \nu \rangle \gtrsim O(E) \), quantum corrections are important.

See J.D. Jackson, Classical Electrodynamics, 3rd edition (John Wiley & Sons, New York, 1998) for more formulas and details. (Note that earlier editions had \( \omega_c \) twice as large as Eq. (7.13).