7. Electromagnetic Relations

Revised September 2005 by H.G. Spieler (LBNL)

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Gaussian CGS</th>
<th>SI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conversion factors</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Charge:</td>
<td>$2.997\ 924\ 58 \times 10^9$ esu</td>
<td>$= 1 \text{ C} = 1 \text{ A s}$</td>
</tr>
<tr>
<td>Potential:</td>
<td>$(1/299.792\ 458)$ statvolt (ersgs/esu)</td>
<td>$= 1 \text{ V} = 1 \text{ J C}^{-1}$</td>
</tr>
<tr>
<td>Magnetic field:</td>
<td>$10^4$ gauss $= 10^4$ dyne/esu</td>
<td>$= 1 \text{ T} = 1 \text{ N A}^{-1} \text{ m}^{-1}$</td>
</tr>
</tbody>
</table>
| F = $q (E + \frac{\mathbf{v}}{c} \times \mathbf{B})$ | F = $q (E + \mathbf{v} \times \mathbf{B})$ | \(\nabla \cdot \mathbf{D} = 4\pi \rho \)
|                  |                    | \(\nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} = \frac{4\pi}{c} \mathbf{J} \)
|                  |                    | \(\nabla \cdot \mathbf{B} = 0 \)
|                  |                    | \(\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \)
| Constitutive relations: | \(\mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{H} = \mathbf{B}/\mu \) | \(\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \quad \mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M} \)
| Linear media:     | \(\mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{H} = \mathbf{B}/\mu \) | \(\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \quad \mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M} \)

\[
\begin{align*}
\mathbf{V} & = \sum_{\text{charges}} \frac{q_i}{r_i} = \int \frac{\rho(r')}{|r-r'|} d^3r', \\
\mathbf{A} & = \int \frac{1}{c} \int \frac{J(r')}{|r-r'|} d^3r'.
\end{align*}
\]

\[
\begin{align*}
\mathbf{E}'_\parallel & = \mathbf{E}_\parallel, \\
\mathbf{E}'_\perp & = \gamma (\mathbf{E}_\perp + \frac{1}{c} \mathbf{v} \times \mathbf{B}), \\
\mathbf{B}'_\parallel & = \mathbf{B}_\parallel, \\
\mathbf{B}'_\perp & = \gamma (\mathbf{B}_\perp - \frac{1}{c} \mathbf{v} \times \mathbf{E}).
\end{align*}
\]

\[
\frac{1}{4\pi \epsilon_0} = c^2 \times 10^{-7} \text{ N} \text{ A}^{-2} = 8.987\ 55 \ldots \times 10^8 \text{ m F}^{-1}; \quad \mu_0 = 4\pi \times 10^{-7} \text{ N} \text{ A}^{-2}; \quad c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 2.997\ 924\ 58 \times 10^8 \text{ m s}^{-1}
\]

7.1 Impedances (SI units)
\(\rho = \text{resistivity at room temperature in } 10^{-8} \Omega \text{ m:}\)
\(~ 1.7 \text{ for Cu} \quad ~ 5.5 \text{ for W} \)
\(~ 2.4 \text{ for Au} \quad ~ 73 \text{ for SS 304} \)
\(~ 2.8 \text{ for Al} \quad ~ 100 \text{ for Nichrome} \)
\(\text{(Al alloys may have double the Al value.)}\)

For alternating currents, instantaneous current \(I\), voltage \(V\), angular frequency \(\omega\):
\[
V = V_0 e^{j\omega t} = ZI. \quad (7.1)
\]

Impedance of self-inductance \(L\): \(Z = j\omega L\).

P.A. Zyla et al. (Particle Data Group), Prog. Theor. Exp. Phys. \textbf{2020}, 083C01 (2021) and 2021 update
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Impedance of capacitance $C$: $Z = 1/j\omega C$.

Impedance of free space: $Z = \sqrt{\mu_0/\epsilon_0} = 376.7$ Ω.

High-frequency surface impedance of a good conductor:

$$Z = \frac{(1 + j)\rho}{\delta}, \text{ where } \delta = \text{skin depth} \quad (7.2)$$

$$\delta = \sqrt{\frac{\rho}{\pi
u\mu}} \approx \frac{6.6 \text{ cm}}{\sqrt{\nu(\text{Hz})}} \text{ for Cu} \quad (7.3)$$

### 7.2 Capacitors, inductors, and transmission lines

The capacitance between two parallel plates of area $A$ spaced by the distance $d$ and enclosing a medium with the dielectric constant $\epsilon$ is

$$C = K\epsilon A/d, \quad (7.4)$$

where the correction factor $K$ depends on the extent of the fringing field. If the dielectric fills the capacitor volume without extending beyond the electrodes, the correction factor $K \approx 0.8$ for capacitors of typical geometry.

The inductance at high frequencies of a straight wire whose length $\ell$ is much greater than the wire diameter $d$ is

$$L \approx 2.0 \left[ \frac{\text{nH}}{\text{cm}} \right] \cdot \ell \left( \ln \left( \frac{4\ell}{d} \right) - 1 \right). \quad (7.5)$$

For very short wires, representative of vias in a printed circuit board, the inductance is

$$L(\text{in nH}) \approx \ell/d. \quad (7.6)$$

A transmission line is a pair of conductors with inductance $L$ and capacitance $C$. The characteristic impedance $Z = \sqrt{L/C}$ and the phase velocity $v_p = 1/\sqrt{LC} = 1/\sqrt{\mu\epsilon}$, which decreases with the inverse square root of the dielectric constant of the medium. Typical coaxial and ribbon cables have a propagation delay of about 5 ns/cm.

The impedance of a coaxial cable with outer diameter $D$ and inner diameter $d$ is

$$Z = 60 \Omega \cdot \frac{1}{\sqrt{\epsilon_r}} \ln \frac{D}{d}, \quad (7.7)$$

where the relative dielectric constant $\epsilon_r = \epsilon/\epsilon_0$. A pair of parallel wires of diameter $d$ and spacing $a > 2.5d$ has the impedance

$$Z = 120 \Omega \cdot \frac{1}{\sqrt{\epsilon_r}} \ln \frac{2a}{d}. \quad (7.8)$$

This yields the impedance of a wire at a spacing $h$ above a ground plane,

$$Z = 60 \Omega \cdot \frac{1}{\sqrt{\epsilon_r}} \ln \frac{4h}{d}. \quad (7.9)$$

A common configuration utilizes a thin rectangular conductor above a ground plane with an intermediate dielectric (microstrip). Detailed calculations for this and other transmission line configurations are given by Gunston [1].

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7.3 Synchrotron radiation (CGS units)
For a particle of charge $e$, velocity $v = \beta c$, and energy $E = \gamma mc^2$, traveling in a circular orbit of radius $R$, the classical energy loss per revolution $\delta E$ is

$$\delta E = \frac{4\pi}{3} \frac{e^2}{R} \beta^3 \gamma^4.$$ (7.10)

For high-energy electrons or positrons ($\beta \approx 1$), this becomes

$$\delta E \text{ (in MeV)} \approx 0.0885 \left[\frac{E \text{ (in GeV)}}{R \text{ (in m)}}\right]^{4/3}.$$ (7.11)

For $\gamma \gg 1$, the energy radiated per revolution into the photon energy interval $d(h\omega)$ is

$$dI = \frac{8\pi}{9} \alpha \gamma F(\omega/\omega_c) \ d(h\omega),$$ (7.12)

where $\alpha = e^2/hc$ is the fine-structure constant and

$$\omega_c = \frac{3\gamma^3 c}{2R}.$$ (7.13)

is the critical frequency. The normalized function $F(y)$ is

$$F(y) = \frac{9}{8\pi} \sqrt{3} y \int_y^{\infty} K_{5/3}(x) \ dx,$$ (7.14)

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where $K_{5/3}(x)$ is a modified Bessel function of the third kind. For electrons or positrons,

$$h\omega_c \text{ (in keV)} \approx 2.22 \left[ E \text{ (in GeV)} \right]^3 / R \text{ (in m)} .$$  \hfill (7.15)

For $\gamma \gg 1$ and $\omega \ll \omega_c$,

$$\frac{dI}{d(h\omega)} \approx 3.3\alpha (\omega R/c)^{1/3} ,$$  \hfill (7.16)

whereas for

$$\gamma \gg 1 \text{ and } \omega \gtrsim 3\omega_c ,$$

$$\frac{dI}{d(h\omega)} \approx \sqrt{\frac{3\pi}{2}} \alpha \gamma \left( \frac{\omega}{\omega_c} \right)^{1/2} e^{-\omega/\omega_c} \left[ 1 + \frac{55}{72} \frac{\omega_c}{\omega} + \ldots \right] .$$  \hfill (7.17)

The radiation is confined to angles $\lesssim 1/\gamma$ relative to the instantaneous direction of motion. For $\gamma \gg 1$, where Eq. (7.12) applies, the mean number of photons emitted per revolution is

$$N_\gamma = \frac{5\pi}{\sqrt{3}} \alpha \gamma ,$$  \hfill (7.18)

and the mean energy per photon is

$$\langle h\omega \rangle = \frac{8}{15\sqrt{3}} h\omega_c .$$  \hfill (7.19)

When $\langle h\omega \rangle \gtrsim O(E)$, quantum corrections are important [2].

References


\footnote{Note that earlier editions of Ref. [2] had $\omega_c$ twice as large as Eq. (7.13).}