67. V_{ud}, V_{us} the Cabibbo Angle, and CKM Unitarity

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The Cabibbo-Kobayashi-Maskawa (CKM) [1,2] three-generation quark mixing matrix written in terms of the Wolfenstein parameters (λ, A, ρ, η) [3] nicely illustrates the orthonormality constraint of unitarity, as well as the central role played by λ.

$$V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)\ . \tag{67.1}$$

That cornerstone parameter is a carryover from the two-generation Cabibbo angle, $\lambda = \sin(\theta_{\text{Cabibbo}}) = V_{us}$. Its value is an important component in tests of CKM unitarity.

Up until 2003, the precise value of λ was controversial, with kaon decays (specifically $K \to \pi e \nu$) branching fractions suggesting [4] $\lambda \simeq 0.220$, while indirect determinations via V_{ud} obtained from nuclear β-decays combined with unitarity preferred a somewhat larger $\lambda \simeq 0.225 - 0.230$. This difference implied a 2 - 2.5 sigma deviation from the first row unitarity requirement

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1, \tag{67.2}$$

a possible hint [5,6] of new physics effects. Below, we describe the current status of V_{ud} and V_{us}, and their implication for the unitarity test in Eq. (67.2). (Since $|V_{ub}|^2 \simeq 1.7 \times 10^{-5}$ is negligibly small, it is ignored in this discussion.) Eq. (67.2) is currently the most stringent test of unitarity in the CKM matrix. As we shall see, it is again showing signs of a possible 2 to 3 sigma inconsistency.

67.1 V_{ud}

Precise values of V_{ud} have been obtained from superallowed nuclear, neutron, and pion beta decays. Currently, the best determination of V_{ud} comes from analysis of a set of 15 precisely measured superallowed nuclear beta-decays [5,6] ($0^+ \to 0^+$ transitions). Measuring their half-lives, t, and Q values gives the decay rate factors, f, which lead to a precise determination of V_{ud} via [7–12]. Based on several decades of dedicated studies, Hardy and Towner recently updated the average [6]

$$|V_{ud}|^2 = 0.97154(22)(54)_{\text{NS}}/(1+\Delta_{\text{R}}^{\text{V}}), \tag{67.3}$$

where Δ_R^V denotes the so-called inner or universal electroweak radiative corrections (RC) to superallowed nuclear beta decays. Note that an additional uncertainty $(54)_{\text{NS}}$ from nuclear structure (NS) [13] has been recently included [6] in that master formula. A dispersion relation (DR) calculational approach [14] to quantum loop corrections, specifically the gamma-W box diagram, gives $\Delta_R^V = 0.02467(22)$. Because of its small uncertainty and more rigorous theoretical footing, we use that value below. A somewhat different approach [15] found $\Delta_R^V = 0.02426(32)$. These recent values are roughly consistent. Both are larger than the 2018 PDG value of 0.02361(38). Implications and possible nuclear physics modifications of those studies are still under scrutiny [13,16]. Nevertheless, currently the 15 most precisely measured superallowed transitions [12] lead to the DR based weighted average of

$$V_{ud} = 0.97373(11)_{\text{exp.,nucl.}}(9)_{\text{RC}}(27)_{\text{NS}}\ \text{(superallowed)}\ , \tag{67.4}$$

which, assuming unitarity, corresponds to the relatively large $\lambda = 0.2277(13)$. This recent determination of V_{ud} has shifted significantly down compared to the 2018 PDG value of 0.97420(21).

In addition, NS uncertainties are now the dominant contribution to the overall uncertainty. Taken at face value, along with current V_{us} determinations (see subsection 67.2), the reduced V_{ud} would seem to violate the first row unitarity requirement and thus suggest the presence of "new physics".

Measurements of the neutron lifetime, τ_n, and the ratio of axial-vector/vector couplings, $g_A \equiv G_A/G_V$, via neutron decay asymmetries combined with the inner radiative corrections can also be used to determine V_{ud} via the precise formula:

$$|V_{ud}|^2 = \frac{5024.7\ \mathrm{s}}{\tau_n(1+3g_A^2)(1+\Delta_R^V)}, \tag{67.5}$$

where Δ_R^V represents the same inner electroweak radiative corrections [8,9] as discussed above.

Using the current published world averages [17],

$$\begin{aligned} \tau_n^{\mathrm{ave}} &= 879.4(6)\ \mathrm{s}\ \ \text{(1.6 PDG scale factor)} \\ g_A^{\mathrm{ave}} &= 1.2756(13),\ \text{(2.6 PDG scale factor)} \end{aligned} \tag{67.6}$$

leads to

$$|V_{ud}| = 0.9737(3)_{\tau_n}(8)_{g_A}(1)_{\mathrm{RC}}, \tag{67.7}$$

for an inner radiative correction of 0.02467(22), while for 0.02426(32) it increases to 0.9739(9). Both central values are similar to the superallowed nuclear beta decay result reported above. Reconciliation with CKM unitarity suggests a shorter neutron lifetime near 878.5 s or a smaller g_A. We note that the most precise recent neutron lifetime update reported [18], $\tau_n = 877.75(34)$ s, has an uncertainty about half as big as the average given in Eq. 67.6. It corresponds to $|V_{ud}| = 0.9746(8)$, a value more in keeping with unitarity expectations; but too large an uncertainty from g_A to be meaningful. Future neutron studies [19] are expected to resolve any current inconsistencies and further reduce the uncertainties in g_A and τ_n making them a potentially better way to determine V_{ud} without the nuclear physics uncertainties.

The PIBETA experiment at PSI measured the very small ($\mathcal{O}(10^{-8})$) branching ratio for $\pi^+ \to \pi^o e^+ \nu_e$ with about ±0.6% precision. Its result gives [20]

$$|V_{ud}| = 0.9739(27)\left[\frac{BR(\pi^+ \to e^+\nu_e(\gamma))}{1.2325\times 10^{-4}}\right]^{\frac{1}{2}}, \tag{67.8}$$

which is normalized using the very precisely measured $BR(\pi^+ \to e^+\nu_e(\gamma)) = 1.2325(23)\times 10^{-4}$ [7], rather than the theoretical branching ratio of $1.2350(2)\times 10^{-4}$, which if used, would increase $|V_{ud}|$ to 0.9749(27). Theoretical uncertainties in pion beta decay are very small [21], leaving open more than an order of magnitude improvement of its experimental branching ratio before theory uncertainties become a problem. Although challenging, improved measurements of pion beta decay currently under discussion would allow this decay mode to compete with superallowed beta decays and future neutron decay efforts for the most precise direct $|V_{ud}|$ determination.

67.2 V_{us}

$|V_{us}|$ may be directly obtained from kaon decays, hyperon decays, and tau decays. Early determinations most often used $K\ell3$ decays:

$$\Gamma_{K\ell 3} = \frac{G_F^2 M_K^5}{192\pi^3} S_{EW}(1+\delta_K^\ell+\delta_{SU2})C^2\,|V_{us}|^2\,f_+^2(0)I_K^\ell. \tag{67.9}$$

Here, ℓ refers to either e or μ, G_F is the Fermi constant, M_K is the kaon mass, S_{EW} is the short-distance radiative correction, δ_K^ℓ is the mode-dependent long-distance radiative correction, $f_+(0)$ is the calculated form factor at zero momentum transfer for the $\ell\nu$ system, and I_K^ℓ is the phase-space integral, which depends on measured semileptonic form factors. For charged kaon decays, δ_{SU2} is the deviation from one of the ratio of $f_+(0)$ for the charged to neutral kaon decay; it is zero for the neutral kaon. C^2 is 1 (1/2) for neutral (charged) kaon decays. Most early determinations of $|V_{us}|$ were based solely on $K \to \pi e\nu$ decays; $K \to \pi\mu\nu$ decays were not used because of large uncertainties in I_K^μ. The experimental measurements are the semileptonic decay widths (based on the semileptonic branching fractions and lifetime) and form factors (allowing calculation of the phase space integrals). Theory is needed for S_{EW}, δ_K^ℓ, δ_{SU2}, and $f_+(0)$.

Many measurements during the last 20 years have resulted in a shift in $|V_{us}|$. Most importantly, the $K \to \pi e\nu$ branching fractions are significantly different than much earlier PDG averages, probably as a result of inadequate treatment of radiation in older experiments. This effect was first observed by BNL E865 [22] in the charged kaon system and then by KTeV [23,24] in the neutral kaon system; subsequent measurements were made by KLOE [25–28], NA48 [29–31], and ISTRA+ [32]. Current averages (*e.g.*, by the PDG [33] or Flavianet [34]) of the semileptonic branching fractions are based only on recent, high-statistics experiments where the treatment of radiation is clear. In addition to measurements of branching fractions, new measurements of lifetimes [35] and form factors [36–40], have resulted in improved precision for all of the experimental inputs to $|V_{us}|$. Precise measurements of form factors for $K_{\mu3}$ decay make it possible to use both semileptonic decay modes to extract V_{us}.

Following the analysis of Moulson [41] and the Flavianet group [34, 42], along with recent improvements in the QED radiative corrections [43], one finds [44], after including the isospin violating effect, δ_{SU2} , the values of $|V_{us}|f_+(0)$ in Table 67.1. The average of these measurements, including correlation effects [44] gives

$$f_+(0)|V_{us}| = 0.21635(38)(3) \tag{67.10}$$

where the errors correspond to Kaon experimental parameters and radiative corrections respectively.

Lattice QCD calculations of $f_+(0)$ have been carried out for 2, 2+1, and 2+1+1 quark flavors and range from about 0.96 to 0.97. Here, we illustrate recent FLAG (2020) updated averages [45] for 2+1 and 2+1+1 flavors:

$$\begin{aligned} f_+(0) &= 0.9677(27) \quad N_f = 2+1 \\ f_+(0) &= 0.9698(17) \quad N_f = 2+1+1 \end{aligned} \tag{67.11}$$

One finds from Eq. (67.10) and Eq. (67.11),

$$\begin{aligned} |V_{us}| &= 0.2236(4)_{\text{exp+RC}}(6)_{\text{lattice}}\ (N_f = 2+1,\ K_{\ell3} \text{ decays}) \\ &= 0.2231(4)_{\text{exp+RC}}(4)_{\text{lattice}}\ (N_f = 2+1+1,\ K_{\ell3} \text{ decays}) \end{aligned} \tag{67.12}$$

A value of V_{us} can also be obtained from a comparison of the radiative inclusive decay rates for $K \to \mu\nu(\gamma)$ and $\pi \to \mu\nu(\gamma)$ combined with a lattice gauge theory calculation of f_{K^+}/f_{π^+} via

$$\frac{|V_{us}|f_{K^+}}{|V_{ud}|f_{\pi^+}} = 0.23871(20)\left[\frac{\Gamma(K\to\mu\nu(\gamma))}{\Gamma(\pi\to\mu\nu(\gamma))}\right]^{\frac{1}{2}} \tag{67.13}$$

with the small error coming from electroweak radiative corrections [46–48]; these corrections were confirmed by direct lattice calculation of the kaon and pion leptonic decay rates [47,48]. Employing

$$\frac{\Gamma(K\to\mu\nu(\gamma))}{\Gamma(\pi\to\mu\nu(\gamma))} = 1.3367(28), \tag{67.14}$$

Table 67.1: $|V_{us}|f_+(0)$ from $K\ell 3$, based on ref. [44]

Decay Mode	$\lvert V_{us}\rvert f_+(0)$
$K^\pm e3$	0.21714 ± 0.00091
$K^\pm \mu 3$	0.21703 ± 0.00114
$K_L e3$	0.21617 ± 0.00047
$K_L \mu 3$	0.21664 ± 0.00058
$K_S e3$	0.21530 ± 0.00122
$K_S \mu 3$	0.21265 ± 0.00467
Average (including correlation effects [44])	0.21635 ± 0.00038

which includes $\Gamma(K \to \mu\nu(\gamma)) = 5.134(11) \times 10^7 s^{-1}$ [41,49], leads to

$$\frac{|V_{us}|f_{K^+}}{|V_{ud}|f_{\pi^+}} = 0.27600(37). \tag{67.15}$$

Employing the FLAG [45] lattice QCD averages for the isospin broken decay constants

$$\begin{aligned}\frac{f_{K^+}}{f_{\pi^+}} &= 1.1917(37) \qquad N_f = 2+1 \\ &= 1.1932(21) \quad N_f = 2+1+1.\end{aligned} \tag{67.16}$$

along with the value of $|V_{ud}|$ in Eq. (67.4) leads to

$$\begin{aligned}|V_{us}| &= 0.2255(8)\ (N_f = 2+1,\ K_{\mu 2}\text{ decays}) \\ &= 0.2252(5)\ (N_f = 2+1+1,\ K_{\mu 2}\text{ decays}).\end{aligned} \tag{67.17}$$

Together, weighted averages of the $K\ell 3$ (Eq. (67.12)) and $K\mu 2$ (Eq. (67.17)) values give similar results for $N_f = 2+1$ and $2+1+1$ flavors:

$$\begin{aligned}|V_{us}| &= 0.2244(5) \quad N_f = 2+1 \\ |V_{us}| &= 0.2243(4) \quad N_f = 2+1+1.\end{aligned} \tag{67.18}$$

Note that the differences between $K\ell 3$ and $K\mu 2$ values for V_{us} differ by 2 and 3 sigma, respectively, for $N_f = 2+1$ and $2+1+1$ flavors. One should therefore scale the uncertainties in Eq. (67.18) accordingly. For that reason, we allow for a scale factor of 2.7 for both 2+1 and 2+1+1 flavors and average the two values. That approximate procedure leads to $|V_{us}| = 0.2243(8)$ which we use when we consider the first row test of CKM unitarity.

It should be mentioned that hyperon decay fits suggest [50]

$$|V_{us}| = 0.2250(27)\ (\text{Hyperon Decays}) \tag{67.19}$$

modulo SU(3) breaking effects that could shift that value up or down. We note that a representative effort [51] that incorporates SU(3) breaking found $V_{us} = 0.226(5)$. Strangeness changing tau decays, averaging both inclusive and exclusive measurements, give [52]

$$|V_{us}| = 0.2221(13)\ (\text{Tau Decays})\ , \tag{67.20}$$

which differs by about 2 sigma from the kaon determination discussed above, and would, if combined with V_{ud} from super-allowed beta decays, lead to a 4 sigma deviation from unitarity. This discrepancy results mainly from the inclusive tau decay results that rely on Finite Energy Sum Rule techniques and assumptions, as well as experimental uncertainties. Recent investigation of that approach suggests a larger value for V_{us}, which is more in accord with other determinations [53].

Employing the values of V_{ud} and V_{us} with an error scale factor of 2 from Eq. (67.4) and the average obtained after rescaling the errors from Eq. (67.18), respectively, leads to the unitarity consistency check

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9985(6)(4). \tag{67.21}$$

where the first error is the uncertainty from $|V_{ud}|^2$ and the second error is the uncertainty from the average $|V_{us}|^2$ from $N_f = 2 + 1 + 1.$ and $N_f = 2 + 1$. One finds about an overall 2 sigma deviation from unitarity. (The deviation increases to 3 sigma if nuclear structure uncertainties are ignored.) That deviation could be due a problem with $|V_{ud}|$ theory (RC or NS), the lattice determination of $f_+(0)$ or new physics.

67.3 CKM Unitarity Constraints

The current 2 sigma experimental disagreement with unitarity, $|V_{ud}|^2+|V_{us}|^2+|V_{ub}|^2 = 0.9985(7)$, still provides strong confirmation of Standard Model radiative corrections (which range between 3-4% depending on the nucleus used) at a high significance level [54]. In addition, it implies constraints on "New Physics" effects at both the tree and quantum loop levels. Those effects could be in the form of contributions to nuclear beta decays, K decays and/or muon decays, with the last of these providing normalization via the muon lifetime [55], which is used to obtain the Fermi constant, $G_\mu = 1.1663787(6) \times 10^{-5}\text{GeV}^{-2}$.

In the following examples, we illustrate the implications of CKM unitarity for (1) exotic muon decays [56] (beyond ordinary muon decay $\mu^+ \to e^+\nu_e\bar{\nu}_\mu$) and (2) new heavy quark mixing V_{uD} [57]. Other examples in the literature [58, 59] include Z_χ boson quantum loop effects, supersymmetry, leptoquarks, compositeness etc.

67.3.1 *Exotic Muon Decays*

If additional lepton flavor violating decays such as $\mu^+ \to e^+\bar{\nu}_e\nu_\mu$ (wrong neutrinos) occur, they would cause confusion in searches for neutrino oscillations at, for example, muon storage rings/neutrino factories or other neutrino sources from muon decays. Calling the rate for all such decays Γ(exotic μ decays), they should be subtracted before the extraction of G_μ and normalization of the CKM matrix. Since that is not done and unitarity works, one has (at one-sided 95% CL)

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 - BR(\text{exotic } \mu \text{ decays}) \geq 0.9975 \tag{67.22}$$

or

$$BR(\text{exotic } \mu \text{ decays}) \leq 0.0025 \ . \tag{67.23}$$

This bound is a factor of 10 better than the direct experimental bound on $\mu^+ \to e^+\bar{\nu}_e\nu_\mu$.

67.3.2 *New Heavy Quark Mixing*

Heavy D quarks naturally occur in fourth quark generation models and some heavy quark "new physics" scenarios such as E_6 grand unification. Their mixing with ordinary quarks gives rise to V_{uD}, which is constrained by unitarity (one sided 95% CL)

$$\begin{gathered} |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 - |V_{uD}|^2 \geq 0.9975 \\ |V_{uD}| \leq 0.05 \ . \end{gathered} \tag{67.24}$$

A similar constraint applies to heavy neutrino mixing and the couplings $V_{\mu N}$ and V_{eN}.

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