

16. Heavy-Quark and Soft-Collinear Effective Theory

Revised August 2023 by C.W. Bauer (LBNL) and M. Neubert (PRISMA, Mainz Inst. for Theor. Physics, JG U.).

16.1 Effective Field Theories

Quantum field theories provide the most precise computational tools for describing physics at the highest energies. One of their characteristic features is that they almost inevitably involve multiple length scales. When trying to determine the value of an observable, quantum field theory demands that all possible virtual states and hence all particles be included in the calculation. Since these particles have widely different masses, the final prediction is sensitive to many scales. This fact represents a formidable challenge from a practical point of view. No realistic quantum field theories can be solved exactly, so that one needs to resort to approximation schemes; these, however, are typically most straightforward when only a single scale is involved at a time.

Effective field theories (EFTs) provide a general theoretical framework to deal with the multi-scale problems of realistic quantum field theories. This framework aims at reducing such problems to a combination of separate and simpler single-scale problems; simultaneously, however, it provides an organization scheme whereby the other scales are not omitted but allowed to play their role in a separate step of the computation. The philosophy and basic principles of this approach are very generic, and correspondingly EFTs represent a widely used method in many different areas of high-energy physics, from the low-energy scales of atomic and nuclear physics to the high-energy scales of (partly yet unknown) elementary-particle physics, see [1–3] for some early references. EFTs can play a role both within analytic perturbative computations and in the context of non-perturbative numerical simulations. One of the simplest applications of EFTs to particle physics concerns the description of an underlying theory that is only probed at energy scales $E < \Lambda$. Any particle with mass $m > \Lambda$ cannot be produced as a real state and therefore only leads to short-distance virtual effects. Thus, one can construct an effective theory in which the quantum fluctuations of such heavy particles are “integrated out” from the generating functional for Green functions. This results in a simpler theory containing only those degrees of freedom that are relevant to the energy scales under consideration. In fact, the standard model of particle physics itself is widely viewed as an EFT of some yet unknown, more fundamental theory [4, 5].

The development of any effective theory starts by identifying the degrees of freedom that are relevant to describe the physics at a given energy (or length) scale and constructing the Lagrangian describing the interactions among these fields. Short-distance quantum fluctuations associated with much smaller length scales are absorbed into the coefficients of the various operators in the effective theory. These coefficients are determined in a matching procedure, by requiring that the EFT reproduces the matrix elements of the full theory up to power corrections. In many cases the effective Lagrangian exhibits enhanced symmetries compared with the fundamental theory, giving non-trivial relations between different observables.

16.2 Heavy-Quark Effective Theory

Heavy-quark systems provide prime examples for applications of the EFT technology, because the hierarchy $m_Q \gg \Lambda_{\text{QCD}}$ (with $Q = b, c$) provides a natural separation of scales. Physics at the scale m_Q is of a short-distance nature and can be treated perturbatively, while for heavy-quark systems there is always also some hadronic physics governed by the confinement scale Λ_{QCD} of the strong interaction. Being able to separate the short-distance and long-distance effects associated with these two scales is crucial for any quantitative description. For instance, if the long-distance hadronic matrix elements are obtained from lattice QCD, then it is necessary to analytically com-

pute the effects of short-wavelength modes that do not fit on the lattice. In many other instances, the long-distance physics can be encoded in a small number of hadronic parameters.

16.2.1 General idea and derivation of the effective Lagrangian

The simplest effective theory for heavy-quark systems is the heavy-quark effective theory (HQET) [6–9] (see [10, 11] for detailed discussions). It provides a simplified description of the soft interactions of a single heavy quark with light partons. This includes the interactions that bind the heavy quark with other light partons inside heavy mesons and baryons.

A softly interacting heavy quark is nearly on-shell. Its momentum may be decomposed as $p_Q = m_Q v + k$, where v is the 4-velocity of the hadron containing the heavy quark. The “residual momentum” k results from the soft interactions of the heavy quark with its environment and satisfies $v \cdot k \sim \Lambda_{\text{QCD}}$ and $k^2 \sim \Lambda_{\text{QCD}}^2$, which in the rest frame of the heavy hadron reduces to $k^\mu \sim \Lambda_{\text{QCD}}$. In the limit $m_Q \gg \Lambda_{\text{QCD}}$, the soft interactions do not change the 4-velocity of the heavy quark, which is therefore a conserved quantum number that is often used as a label on the effective heavy-quark fields.

A nearly on-shell Dirac spinor has two large and two small components. We define

$$Q(x) = e^{-im_Q v \cdot x} [h_v(x) + H_v(x)], \quad (16.1)$$

where

$$h_v(x) = e^{im_Q v \cdot x} \frac{1 + \not{v}}{2} Q(x), \quad H_v(x) = e^{im_Q v \cdot x} \frac{1 - \not{v}}{2} Q(x) \quad (16.2)$$

are the large (“upper”) and small (“lower”) components of the spinor field, respectively. The extraction of the phase factor in (16.1) implies that the fields h_v and H_v carry the residual momentum k . The field H_v is $1/m_Q$ suppressed relative to h_v and describes quantum fluctuations far off the mass shell. Integrating it out using its equations of motion yields the HQET Lagrangian

$$\mathcal{L}_{\text{HQET}} = \bar{h}_v i v \cdot D_s h_v + \frac{1}{2m_Q} \left[\bar{h}_v (iD_s)^2 h_v + C_{\text{mag}}(\mu) \frac{g}{2} \bar{h}_v \sigma_{\mu\nu} G_s^{\mu\nu} h_v \right] + \dots \quad (16.3)$$

The covariant derivative $iD_s^\mu = i\partial^\mu + gA_s^\mu$ and the field strength $G_s^{\mu\nu}$ contain only the soft gluon field. Hard gluons have been integrated out, and their effects are contained in the Wilson coefficients of the operators in the effective Lagrangian. From the leading operator one derives the Feynman rules of HQET. The new operators entering at subleading order are referred to as the “kinetic energy” and “chromo-magnetic interaction”. The kinetic-energy operator corresponds to the first correction term in the Taylor expansion of the relativistic energy $E = m_Q + \vec{p}^2/2m_Q + \dots$. Lorentz invariance, which is encoded as a reparametrization invariance of the effective Lagrangian [12], ensures that its Wilson coefficient is not renormalized ($C_{\text{kin}} \equiv 1$). The coefficient C_{mag} of the chromo-magnetic operator receives corrections starting at one-loop order [6].

16.2.2 Spin-flavor symmetry

The leading term in the HQET Lagrangian exhibits a global spin-flavor symmetry. Its physical meaning is that, in the infinite mass limit, the properties of hadronic systems containing a single heavy quark are insensitive to the spin and flavor of the heavy quark [13, 14]. The spin symmetry results from the fact that there are no Dirac matrices in the leading term of the effective Lagrangian in (16.3), implying that the interactions of the heavy quark with soft gluons leave its spin unchanged. The flavor symmetry arises since the mass of the heavy quark does not appear at leading order. For n_Q heavy quarks moving at the same velocity, one can simply extend (16.3) by summing over n_Q identical terms for heavy-quark fields h_v^i . The result is invariant under rotations in flavor space. When combined with the spin symmetry, the symmetry group becomes promoted to $SU(2n_Q)$. These symmetries are broken by the operators at subleading power in the $1/m_Q$ expansion.

The spin-flavor symmetry leads to many interesting relations between the properties of hadrons containing a heavy quark. The most direct consequences concern the spectroscopy of such states [15]. In the heavy-quark limit, the spin of the heavy quark and the total angular momentum j of the light degrees of freedom are separately conserved by the strong interactions. Because of heavy-quark symmetry, the dynamics is independent of the spin and mass of the heavy quark. Hadronic states can thus be classified by the quantum numbers (flavor, spin, parity, etc.) of the light degrees of freedom. The spin symmetry predicts that, for fixed $j \neq 0$, there is a doublet of degenerate states with total spin $J = j \pm \frac{1}{2}$. The flavor symmetry relates the properties of states with different heavy-quark flavor.

16.2.3 Weak decay form factors

Of particular interest are the relations between the weak decay form factors of heavy mesons, which parametrize hadronic matrix elements of currents between two mesons containing a heavy quark. These relations have been derived by Isgur and Wise [14], generalizing ideas developed by Nussinov and Wetzel [16] and Voloshin and Shifman [17]. For the purpose of this discussion, it is convenient to work with a mass-independent normalization of meson states and use velocity rather than momentum variables.

Consider the elastic scattering of a pseudoscalar meson, $P(v) \rightarrow P(v')$, induced by an external vector current coupled to the heavy quark contained in P , which acts as a color source moving with the meson's velocity v . The action of the current is to replace instantaneously the color source by one moving at velocity v' . Soft gluons need to be exchanged in order to rearrange the light degrees of freedom and build up the final state meson moving at velocity v' . This rearrangement leads to a form-factor suppression. The important observation is that, in the $m_Q \rightarrow \infty$ limit, the form factor can only depend on the Lorentz boost $\gamma = v \cdot v'$ connecting the rest frames of the initial and final-state mesons (as long as $\gamma = \mathcal{O}(1)$). In the effective theory the hadronic matrix element describing the scattering process can therefore be written as

$$\langle P(v') | \bar{h}_{v'} \gamma^\mu h_v | P(v) \rangle = \xi(v \cdot v') (v + v')^\mu, \quad (16.4)$$

with a form factor $\xi(v \cdot v')$ that is real and independent of m_Q . By flavor symmetry, the form factor remains identical when one replaces the heavy quark Q in one of the meson states by a heavy quark Q' of a different flavor, thereby turning P into another pseudoscalar meson P' . At the same time, the current becomes a flavor-changing vector current. This universal form factor is called the Isgur-Wise function [14]. For equal velocities the vector current $J^\mu = \bar{h}_v \gamma^\mu h_v$ is conserved in the effective theory, irrespective of the flavor of the heavy quarks. The corresponding conserved charges are the generators of the flavor symmetry. It follows that the Isgur-Wise function is normalized at the point of equal velocities: $\xi(1) = 1$. Since the recoil energy of the daughter meson P' in the rest frame of the parent meson P is $E_{\text{recoil}} = m_{P'} (v \cdot v' - 1)$, the point $v \cdot v' = 1$ is referred to as the zero-recoil limit. The heavy-quark spin symmetry leads to additional relations among weak decay form factors. It can be used to relate matrix elements involving vector mesons to those involving pseudoscalar mesons, which once again can be described completely in terms of the universal Isgur-Wise function.

The form factor relations imposed by heavy-quark symmetry describe the semileptonic decay processes $\bar{B} \rightarrow D \ell \bar{\nu}$ and $\bar{B} \rightarrow D^* \ell \bar{\nu}$ in the limit of infinite heavy-quark masses. They are model-independent consequences of QCD. The known normalization of the Isgur-Wise function at zero recoil can be used to obtain a model-independent measurement of the element $|V_{cb}|$ of the Cabibbo-Kobayashi-Maskawa (CKM) matrix. The semileptonic decay $\bar{B} \rightarrow D^* \ell \bar{\nu}$ is particularly well suited for this purpose [18]. Experimentally this is a very clean mode, since the reconstruction of the D^* meson mass provides a powerful rejection against background. From the theoretical point of

view, it is ideal since the decay rate at zero recoil is protected by Luke's theorem against first-order power corrections in $1/m_Q$ [19]. This is described in more detail in Section 12. Corrections to the heavy-quark symmetry relations for the $\bar{B} \rightarrow D^{(*)}$ form factors near zero recoil can also be constrained using sum rules derived in the small-velocity limit [20, 21].

16.2.4 Decoupling transformation

At leading order in $1/m_Q$, the couplings of soft gluons to heavy quarks in the effective Lagrangian (16.3) can be removed by the field redefinition $h_v(x) = Y_v(x) h_v^{(0)}(x)$, where $Y_v(x)$ is a soft Wilson line along the direction of v , extending from minus infinity to the point x . In terms of the new fields the leading-order HQET Lagrangian becomes $\mathcal{L}_{\text{HQET}} = \bar{h}_v^{(0)} i v \cdot \partial h_v^{(0)}$. It describes a free theory as far as the strong interactions of heavy quarks are concerned. However, the theory is nevertheless non-trivial in the presence of external sources. Consider, e.g., the case of a weak-interaction heavy-quark current

$$\bar{h}_{v'} \gamma^\mu (1 - \gamma_5) h_v = \bar{h}_{v'}^{(0)} \gamma^\mu (1 - \gamma_5) Y_{v'}^\dagger Y_v h_v^{(0)}, \quad (16.5)$$

where v and v' are the velocities of the heavy mesons containing the heavy quarks. Unless the two velocities are equal, corresponding to the zero-recoil limit discussed above, the object $Y_{v'}^\dagger Y_v$ is non-trivial, and hence the soft gluons do not decouple from the heavy quarks inside the current operator. One may interpret $Y_{v'}^\dagger Y_v$ as a Wilson loop with a cusp at the point x , where the two paths parallel to the different velocity vectors intersect. The presence of the cusp leads to non-trivial ultra-violet behavior (for $v \neq v'$), which is described by a cusp anomalous dimension $\Gamma_{\text{cusp}}(v \cdot v')$ that was calculated at two-loop order in [22]. It coincides with the velocity-dependent anomalous dimension of heavy-quark currents, which was introduced in the context of HQET in [23]. The interpretation of heavy quarks as Wilson lines is a useful tool, which was put forward in one of the very first papers on the subject [6]. This technology will be useful in the study of the interactions of heavy quarks with collinear degrees of freedom discussed later in this review.

16.2.5 Heavy-quark expansion for inclusive decays

The theoretical description of inclusive decays of hadrons containing a heavy quark exploits two observations [24–28]: bound-state effects related to the initial state can be calculated using the heavy-quark expansion, and the fact that the final state consists of a sum over many hadronic channels eliminates the sensitivity to the properties of individual final-state hadrons. The second feature rests on the hypothesis of quark-hadron duality, i.e. the assumption that decay rates are calculable in QCD after a smearing procedure has been applied [29]. In semileptonic decays, the integration over the lepton spectrum provides a smearing over the invariant hadronic mass of the final state (global duality). For nonleptonic decays, where the total hadronic mass is fixed, the summation over many hadronic final states provides an averaging (local duality). Since global duality is a much weaker assumption, the theoretical control of inclusive semileptonic decays is on firmer footing.

Using the optical theorem, the inclusive decay width of a hadron H_b containing a b quark can be written in the form

$$\Gamma(H_b) = \frac{1}{M_{H_b}} \text{Im} \langle H_b | i \int d^4x T \{ \mathcal{H}_{\text{eff}}(x), \mathcal{H}_{\text{eff}}(0) \} | H_b \rangle. \quad (16.6)$$

The effective weak Hamiltonian for b -quark decays consists of dimension-6 four-fermion operators and dipole operators [30]. Because of the large mass of the b quark, it follows that the separation of fields in the time-ordered product in (16.6) is small, of order $x \sim 1/m_b$. It is thus possible to construct an operator-product expansion (OPE) for the time-ordered product, in which it is represented as a series of local operators in HQET. The leading operator $\bar{h}_v h_v$ has a trivial matrix

element. The next contributions arise at $\mathcal{O}(1/m_b^2)$ and give rise to two parameters $\mu_\pi^2(H_b)$ and $\mu_G^2(H_b)$, which are defined as the matrix elements of the heavy-quark kinetic energy and chromomagnetic interaction inside the hadron H_b , respectively [31]. For the ground-state heavy mesons and baryons, one has $\mu_G^2(B) = 3(m_{B^*}^2 - m_B^2)/4 \simeq 0.36 \text{ GeV}^2$ and $\mu_G^2(\Lambda_b) = 0$. Thus, the total inclusive decay rate of a hadron H_b can be written as [25, 26]

$$\Gamma(H_b) = \frac{G_F^2 m_b^5 |V_{cb}|^2}{192\pi^3} \left[c_1 + c_2 \frac{\mu_\pi^2(H_b)}{2m_b^2} + c_3 \frac{\mu_G^2(H_b)}{2m_b^2} + \mathcal{O}\left(\frac{1}{m_b^3}\right) + \dots \right], \quad (16.7)$$

where the prefactor arises from the loop integrations and is proportional to the fifth power of the b -quark mass. The coefficient functions c_i are calculable order by order in perturbation theory. While c_1 corresponds to the decay rate of a free heavy quark, the higher-order coefficients systematically account for bound-state effects. The coefficients of the subleading operators and of the leading operator at third order in $1/m_b$ have been calculated at NLO [32–36]. At tree level the heavy-quark expansion has been performed up to fifth order in $1/m_b$ [37].

From the fully inclusive width in (16.7) one can obtain the lifetime of a heavy hadron via $\tau(H_b) = 1/\Gamma(H_b)$. Due to the universality of the leading term in the heavy-quark expansion, lifetime ratios such as $\tau(B^-)/\tau(\bar{B}^0)$, $\tau(\bar{B}_s^0)/\tau(\bar{B}^0)$ and $\tau(\Lambda_b)/\tau(\bar{B}^0)$ are particularly sensitive to the hadronic parameters determining the power corrections in the expansion. In order to understand these ratios theoretically, it is necessary to include phase-space enhanced power corrections of order $(\Lambda_{\text{QCD}}/m_b)^3$ [38, 39] as well as short-distance perturbative effects [40] in the calculation (see [41] for a discussion of the corresponding calculations).

A formula analogous to (16.7) can be derived for differential distributions in specific inclusive decay processes, assuming that these distributions are integrated over a sufficiently large region of phase space to ensure quark-hadron duality. Important examples are the distributions in the lepton energy and the lepton invariant mass, as well as moments of the invariant hadronic mass distribution in the semileptonic processes $\bar{B} \rightarrow X_u \ell \bar{\nu}$ and $\bar{B} \rightarrow X_c \ell \bar{\nu}$. A global fit of semileptonic decay distributions can be used to determine the CKM matrix elements $|V_{ub}|$ and $|V_{cb}|$ along with heavy-quark parameters such as the masses m_b , m_c and the hadronic parameters $\mu_\pi^2(B)$, $\mu_G^2(B)$. These determinations provide some of the most accurate values for these parameters (see e.g. [42–45]).

16.2.6 Shape functions and non-local power corrections

In certain regions of phase space, in which the hadronic final state in an inclusive heavy-hadron decay is made up of light energetic partons, the local OPE for inclusive decays must be replaced by a more complicated expansion involving hadronic matrix elements of non-local light-ray operators [46, 47]. Prominent examples are the radiative decay $\bar{B} \rightarrow X_s \gamma$ for large photon energy E_γ near $m_B/2$, and the semileptonic decay $\bar{B} \rightarrow X_u \ell \bar{\nu}$ at large lepton energy or small hadronic invariant mass. In these cases, the differential decay rates at leading order in the heavy-quark expansion can be written in the factorized form $d\Gamma = H J \otimes S$ [48], where the hard function H and the jet function J are calculable in perturbation theory. The characteristic scales for these functions are set by m_b and $(m_b \Lambda_{\text{QCD}})^{1/2}$, respectively. The soft function

$$S(\omega) = \int \frac{dt}{4\pi} e^{-i\omega t} \langle \bar{B}(v) | \bar{h}_v(tn) Y_n(tn) Y_n^\dagger(0) h_v(0) | \bar{B}(v) \rangle \quad (16.8)$$

is a non-perturbative object called the shape function [46, 47]. Here Y_n are soft Wilson lines along a light-like direction n aligned with the momentum of the hadronic final-state jet. The jet function and the shape function share a common variable $\omega \sim \Lambda_{\text{QCD}}$, and the symbol $\otimes$ denotes a convolution in this variable.

While the hard functions are different for the decays $\bar{B} \rightarrow X_s \gamma$ and $\bar{B} \rightarrow X_u \ell \bar{\nu}$, the jet and soft functions are identical at leading order in Λ_{QCD}/m_Q . This is particularly important for the shape function, which introduces non-perturbative physics into the theoretical predictions for the decay rates in the regions of experimental interest. The fact that both processes depend on the same non-perturbative function makes it possible to use the measured shape of the $\bar{B} \rightarrow X_s \gamma$ photon spectrum to reduce the theoretical uncertainties in the determination of the CKM element $|V_{ub}|$ from semileptonic decays. In higher orders of the heavy-quark expansion, an increasing number of subleading jet and soft functions are required to describe the decay distributions [49]. These have been analyzed in detail at order $1/m_b$ [50–52]. In the case of $\bar{B} \rightarrow X_s \gamma$ (and also in the related case of $\bar{B} \rightarrow X_s \ell \bar{\ell}$), some of these non-local effects survive in the total decay rate and give rise to irreducible hadronic uncertainties [53]. The technology for deriving the corresponding factorization theorems relies on the soft-collinear effective theory, to which we now turn.

16.3 Soft-Collinear Effective Theory

As discussed in the previous section, soft gluons that bind a heavy quark inside a heavy meson cannot change the virtuality of that heavy quark by a significant amount. The ratio Λ_{QCD}/m_Q provides the expansion parameter in HQET, which is a small parameter since $m_Q \gg \Lambda_{\text{QCD}}$. This obviously does not work when considering light quarks. However, if the energy Q (not to be confused with the flavor of a heavy quark) of the quarks is large, the ratio Λ_{QCD}/Q provides a small parameter, which can be used to construct an effective theory. One major difference to HQET is that light energetic quarks cannot only emit soft gluons, but they can also emit collinear gluons (an energetic gluon in the same direction as the original quark), without parametrically changing their virtuality. Thus, to fully reproduce the long-distance physics of energetic quarks requires that one includes their interactions with both soft and collinear particles. The resulting effective theory is therefore called soft-collinear effective theory (SCET) [54–57] (see [58] for a review).

A single energetic particle can always be boosted to a frame where all momentum components have similar size, in which case there is no small expansion parameter. Thus, the presence of energetic particles must refer to a reference frame defined by external kinematics. SCET has a wide range of applications; some examples are the production of energetic, light states in the decay of a heavy particle in its rest frame, the production of energetic jets in collider environments, and the scattering of energetic particles off a target at rest. In this brief review we will outline the main features of this effective theory and mention a few selected applications.

16.3.1 General idea of the expansion

Consider a quark with virtuality much less than its energy Q , moving along the direction $\vec{n}$. It is convenient to parameterize the momentum p_n of this particle in terms of its light-cone components, defined by $(p_n^-, p_n^+, p_n^\perp) = (\bar{n} \cdot p_n, n \cdot p_n, p_n^\perp)$, where $n^\mu = (1, \vec{n})$ and $\bar{n}^\mu = (1, -\vec{n})$ are light-like reference vectors, and $n \cdot p_n^\perp = \bar{n} \cdot p_n^\perp = 0$. The subscript n on the momentum indicates the direction of the collinear particle. In terms of these light-cone components, the virtuality satisfies $p_n^2 = p_n^+ p_n^- + p_n^{\perp 2}$. The individual components of the momentum obey

$$(p_n^-, p_n^+, p_n^\perp) \sim Q(1, \lambda^2, \lambda), \quad (16.9)$$

where $\lambda^2 = p^2/Q^2$ is the expansion parameter of SCET. The virtuality of such an energetic particle remains parametrically unchanged if it interacts with energetic particles in the same direction n , or with soft particles with momentum scaling as

$$(p_s^-, p_s^+, p_s^\perp) \sim Q(\lambda^2, \lambda^2, \lambda^2). \quad (16.10)$$

SCET is constructed in such a way as to reproduce the long-distance dynamics arising from the interactions of collinear and soft degrees of freedom.

In the above power counting the transverse momenta of soft degrees of freedom scale as $p_s^\perp \sim Q\lambda^2$, which is much smaller than the transverse momenta $p_c^\perp \sim Q\lambda$ of collinear fields. This theory is usually called SCET_I. If the external kinematics require that the transverse momenta of both soft and collinear fields are of the same size, $p_c^\perp \sim p_s^\perp$, then the appropriate degrees of freedom have the scaling $p_c \sim Q(1, \lambda^2, \lambda)$ and $p_s \sim Q(\lambda, \lambda, \lambda)$. This theory is usually called SCET_{II} and is required, e.g., for exclusive hadronic decays such as $\bar{B} \rightarrow D\pi$, where the virtuality of both collinear and soft degrees of freedom are set by Λ_{QCD} , or for the description of transverse-momentum distributions at colliders. SCET_I power counting is assumed in the following sections, while SCET_{II} is discussed in more detail in 16.3.6.

16.3.2 Leading-order Lagrangian

The derivation of the SCET Lagrangian follows similar steps as described for HQET in Section 16.2.1. One begins by deriving the Lagrangian for a theory containing only a single collinear sector. Similar to HQET, one separates the full QCD field into two components, $q_n(x) = \psi_n(x) + \eta_n(x)$, where (with $n \cdot \bar{n} = 2$)

$$\psi_n(x) = \frac{\not{n}\not{\bar{n}}}{4} q_n(x), \quad \eta_n(x) = \frac{\not{\bar{n}}\not{n}}{4} q_n(x). \quad (16.11)$$

The degrees of freedom described by the field η_n are far off shell and can therefore be eliminated using its equation of motion. This gives

$$\mathcal{L}_n = \bar{\psi}_n(x) \left[in \cdot D + i\not{D}^\perp \frac{1}{i\bar{n} \cdot D} i\not{D}^\perp \right] \frac{\not{n}}{2} \psi_n(x). \quad (16.12)$$

As a next step, one separates the large and small momentum components by decomposing the collinear momentum into a “label” and a residual momentum, $p^\mu = P^\mu + k^\mu$ with $n \cdot P = 0$. One then performs a phase redefinition on the collinear fields, such that $\psi_n(x) = e^{-iP \cdot x} \xi_n(x)$. Derivatives acting on the fields $\xi_n(x)$ now only pick out the residual momentum. Since unlike in HQET the label momentum in SCET is not conserved, one defines a label operator $\mathcal{P}^\mu$ acting as $\mathcal{P}^\mu \xi_n(x) = P^\mu \xi_n(x)$ [56], as well as a corresponding covariant label operator $i\mathcal{D}_n^\mu = \mathcal{P}^\mu + gA_n^\mu(x)$. Note that at leading order in power counting $i\mathcal{D}_n^\mu$ does not contain the soft gluon field. This leads to the final SCET Lagrangian [56, 57, 59, 60]

$$\mathcal{L}_n = \bar{\xi}_n(x) \left[in \cdot D_n + gn \cdot A_s + i\not{D}_n^\perp \frac{1}{i\bar{n} \cdot \mathcal{D}_n} i\not{D}_n^\perp \right] \frac{\not{n}}{2} \xi_n(x) + \dots, \quad (16.13)$$

where we have split $in \cdot D$ into a collinear piece $in \cdot D_n = in \cdot \partial + gn \cdot A_n$ and a soft piece $gn \cdot A_s$. This latter term gives rise to the only interaction between a collinear quark and soft gluons at leading power in λ . The ellipses represent higher-order interactions between soft and collinear particles.

The Lagrangian describing collinear fields in different light-like directions is simply given by the sum of the Lagrangians for each direction n , i.e. $\mathcal{L} = \sum_n \mathcal{L}_n$. The soft gluons are the same in each individual Lagrangian. An alternative way to understand the separation between large and small momentum components is to derive the Lagrangian of SCET in position space [60]. In this case no label operators are required, but the collinear field operators are separated by a light-like distance. An important difference between SCET and HQET is that the SCET Lagrangian is not corrected by short distance fluctuations. The physical reason is that in the construction described above no high-momentum modes have been integrated out [60]. Such hard modes arise when different collinear sectors are coupled via some external current (e.g. in jet production at e^+e^- or hadron colliders), or when collinear particles are produced in the rest frame of a decaying heavy object

(such as in B decays). Short-distance effects are then incorporated in the Wilson coefficients of the external source operators.

16.3.3 Collinear gauge invariance and Wilson lines

An important aspect of SCET concerns the implementation of local gauge invariance. Because the effective field operators describe modes with certain momentum scalings, the effective Lagrangian respects only residual gauge symmetries. One of them satisfies the collinear scaling

$$(\bar{n} \cdot \partial, n \cdot \partial, \partial^\perp) U_n(x) \sim Q(1, \lambda^2, \lambda) U_n(x), \quad (16.14)$$

and one the soft scaling

$$(\bar{n} \cdot \partial, n \cdot \partial, \partial^\perp) U_s(x) \sim Q(\lambda^2, \lambda^2, \lambda^2) U_s(x). \quad (16.15)$$

While the soft gauge transformation is common for all fields, collinear fields in different directions each transform under their own collinear gauge transformation. In other words, each collinear sector has to be separately gauge invariant under its collinear gauge transformation. This requires the introduction of collinear Wilson lines [56]

$$W_n(x) = \text{P exp} \left[-ig \int_{-\infty}^0 ds \bar{n} \cdot A_n(s\bar{n} + x) \right], \quad (16.16)$$

which transform under collinear gauge transformations according to $W_n \rightarrow U_n W_n$. Thus, the combination $\chi_n \equiv W_n^\dagger \psi_n$ is gauge invariant. In a similar manner, one can define the gauge-invariant gluon field $B_n^\mu = g^{-1} W_n^\dagger i D_n^\mu W_n$ [61, 62]. Collinear operators in SCET are typically constructed from such collinearly gauge-invariant building blocks.

16.3.4 Derivation of factorization theorems

One of the important applications of SCET is to understand how to factorize cross sections or decay rates involving energetic particles moving in different directions into simpler quantities that can either be calculated perturbatively or determined from data. For the purpose of this discussion, we focus on factorization in the context of collider physics. Note that factorization theorems have been around for much longer than SCET (see [63] for a review). However, the effective theory allows for a conceptually simpler understanding of certain classes of factorization theorems [61], since most simplifications happen already at the level of the Lagrangian. The discussion in this section is valid to leading order in the power counting of the effective theory.

As discussed in the previous section, the Lagrangian of SCET does not involve any direct couplings between collinear particles moving in different directions. A process involving highly energetic initial- and/or final-state particles moving along different directions n_i is therefore described in SCET by operators involving gauge-invariant collinear building blocks for each direction, which only interact amongst each other via soft gluon exchange. At leading power, soft gluons couple to collinear quarks only through the term $\bar{\xi}_n g n \cdot A_s (\not{n}/2) \xi_n$ in the effective Lagrangian in (16.13), which is similar to the coupling of soft gluons to heavy quarks in HQET, see Section 16.2.4. This coupling, and the corresponding coupling to collinear gluons can be removed by means of the field redefinitions [57]

$$\psi_n(x) = Y_n(x) \psi_n^{(0)}(x), \quad A_n^a(x) = Y_n^{ab}(x) A_n^{b(0)}(x), \quad (16.17)$$

where the soft Wilson lines Y_n and Y_n^{ab} live in the fundamental and adjoint representations of $SU(3)$, respectively. Each collinear building block gives rise to a soft Wilson line. After this decoupling, the different collinear sectors no longer interact with each other and the soft interactions are entirely encoded in the product of soft Wilson lines. This is the basis of factorization proofs in SCET.

The simplest applications of this formalism concern inclusive processes, such as the Drell-Yan production of electroweak bosons in the threshold region, $pp \rightarrow Z/H + X_s$, where X_s is a final state containing only soft particles, and event shape distributions in e^+e^- collisions in the two-jet limit. For Drell-Yan the factorization formula is given by [61, 64]

$$\begin{aligned}\sigma(pp \rightarrow ZX_s) &= \sum_q H_q(\mu) [f_{q/p}(\mu) f_{\bar{q}/p}(\mu)] \otimes S_F(\mu), \\ \sigma(pp \rightarrow HX_s) &= H_g(\mu) [f_{g/p}(\mu) f_{g/p}(\mu)] \otimes S_A(\mu),\end{aligned}\tag{16.18}$$

where the collinear matrix elements give rise to the parton distributions $f_{a/p}(\mu)$, the soft function are vacuum matrix element of the soft Wilson lines in the fundamental or adjoint representation, and the hard functions are the squares of the Wilson coefficients of the relevant SCET operators. The convolution symbol indicates a convolution over the longitudinal momentum fractions of the colliding partons. The objects in the factorization theorem depend on the renormalization scale μ , whereas the cross sections are scale independent. Solving the renormalization group equations (RGE) allows to resum large logarithms in the cross section (see below).

For event shapes such as thrust T , no initial-state partons are present, and the factorization formula takes the form (with $\tau = 1 - T$) [65, 66]

$$\frac{d\sigma}{d\tau} = H(\mu) [J_n(\mu) J_{\bar{n}}(\mu)] \otimes S_{n\bar{n}}(\mu),\tag{16.19}$$

where the the jet functions are vacuum matrix elements of the outgoing collinear quark fields, which are convoluted with the soft function $S_{n\bar{n}}$ defined as another vacuum matrix element of the soft Wilson lines.

For a more general high-energy scattering process such as $pp \rightarrow N$ jets, where the incoming protons move along directions $n_{a,b}$ and the jets define directions $n_1, \dots, n_N$, the factorization theorem for the differential cross section can be written as

$$d\sigma \sim \sum_{ab} H_{ab\dots N}(\mu) [B_{a/p}(\mu) B_{b/p}(\mu)] \otimes [J_1(\mu) \dots J_N(\mu)] \otimes S_{ab\dots N}(\mu).\tag{16.20}$$

where the beam functions are generalized parton distributions [67] and the jet functions depend on the algorithm used to define the jets. It should be mentioned that the most difficult part of traditional factorization proofs involves showing that so-called Glauber gluons do not spoil the above factorization theorem [68]. The description of Glauber effects within SCET [69] has been developed in [70], where a closed form for the effective Lagrangian describing these interactions was derived. The formalism has been extended to Glauber quarks in [71]. In this context, a proof of factorization requires demonstrating that this Lagrangian has no impact on a particular cross section. If an observable is non-global, in the sense that energetic radiation is restricted to certain regions in phase space, the above factorization theorem needs to be generalized [72–74].

16.3.5 Resummation of large logarithms

Many processes of interest depend on multiple scales which can be widely separated. The perturbative series of such processes depends logarithmically on large scale ratios. For high-energy processes involving soft and collinear particles, there can be up to two powers of large logarithms for each power of the coupling constant. These double logarithms are referred to as Sudakov logarithms and can spoil the perturbative expansion.

SCET can be used to sum large logarithms L using renormalization-group equations. This reorganizes the perturbative series in such a way that the product $\alpha_s L$ is treated as an $\mathcal{O}(1)$

parameter. In many cases, one can write a perturbative series for the logarithm of a cross section, such that

$$\sigma \sim \exp [L g_0(\alpha_s L) + g_1(\alpha_s L) + \alpha_s g_2(\alpha_s L) + \dots] , \quad (16.21)$$

with functions $g_n(x)$ that need to be determined. Note that each term in this series in the exponent is one power of α_s suppressed relative to the previous term.

The important ingredient in achieving this resummation is the fact that SCET factorizes a given cross section into simpler pieces, each of which depends on a single physical scale. The only dependence on that scale can arise through logarithms of its ratio with the renormalization scale μ . Thus, for each of the components in the factorization theorem one can choose a renormalization scale μ for which the large logarithmic terms are absent. Of course, the factorization formula requires a common renormalization scale μ in all its components, and one therefore has to use the renormalization group (RG) to evolve the various component functions from their preferred scale to the common scale μ . A novel feature of RG equations in SCET, as opposed to other EFTs, is that the anomalous dimensions entering the evolution equations of the hard, beam, jet and soft functions in a factorization formula such as (16.20) contain a single power of the logarithm of the relevant energy scale. For example, the anomalous dimension γ_H of the hard function has the form

$$\gamma_H(\mu) = c_H \Gamma_{\text{cusp}}(\alpha_s) \ln \frac{Q^2}{\mu^2} + \gamma(\alpha_s) , \quad (16.22)$$

where c_H is a process-dependent coefficient and Γ_{cusp} denotes the so-called cusp anomalous dimension [22, 75]. Collinear and soft functions have similar anomalous dimensions, which also involve a cusp and a non-cusp part. The non-cusp part γ of the anomalous dimensions is process (and observable) dependent. The presence of a logarithm in the anomalous dimension is characteristic of Sudakov problems and arises since the perturbative series contains double logarithms of scale ratios.

Solving the RG equations one can systematically resum all large logarithms of scale ratios in the factorized cross section and express the functions $g_n(\alpha_s L)$ introduced above in terms of ratios of running coupling constants. In order to compute the first two terms $L g_0(\alpha_s L) + g_1(\alpha_s L)$ in $\ln \sigma$, corresponding to the next-to-leading logarithmic (NLL) approximation, one needs two-loop expressions for the cusp anomalous dimension and β function, one-loop expressions for the non-cusp pieces in the anomalous dimensions, and tree-level matching conditions for all component functions at their characteristic scales. To calculate the next term $\alpha_s g_2(\alpha_s L)$ in the expansion, corresponding to NNLL order, one needs to go one order higher in the loop expansion, and so on.

16.3.6 Factorization and resummation in SCET_{II}

The effective theory SCET_{II} contains collinear and soft particles with momenta scaling as $(p_n^-, p_n^+, p_n^\perp) \sim Q(1, \lambda^2, \lambda)$ and $(p_s^-, p_s^+, p_s^\perp) \sim Q(\lambda, \lambda, \lambda)$. They have the same small virtuality ($p_n^2 \sim p_s^2 \sim Q^2 \lambda^2$) but differ in their rapidities. An important class of observables, for which this scaling is relevant, contains cross sections for processes in which the transverse momenta of particles are constrained by external kinematics. The prime example are the transverse-momentum distributions of electroweak gauge bosons or Higgs bosons produced at hadron colliders. The parton transverse momenta are constrained by the fact that their vector sum must be equal and opposite to the transverse momentum q_T of the boson. Standard RG evolution in the effective theory controls the logarithms arising from the fact that the virtualities of the collinear and soft modes are much smaller than the hard scale Q in the process (the boson mass). However, additional large logarithms arise since the rapidities of collinear and soft modes are parametrically different, such that $e^{|y_c - y_s|} \sim 1/\lambda$. These logarithms can be traced to a new source of divergences and an unusual

failure of dimensional regularization. They need to be factorized in the cross section and resummed by other means.

Two approaches exist for how to deal with the additional rapidity logarithms in SCET. In the approach of [76], they are interpreted as a consequence of a “collinear anomaly” of the effective theory SCET_{II}, resulting from the fact that a classical rescaling symmetry of the effective Lagrangian is broken by quantum effects. The extra large logarithms can be resummed by means of simple differential equations, which typically state that, to all orders in perturbation theory (and in an appropriate space), the logarithm of the cross section contains only a single extra logarithm of $\lambda \sim q_T/Q$ not contained in the hard function. Another approach to resum the rapidity logarithms uses the “rapidity renormalization group” [77], in which the relevant differential equations are obtained by considering a new type of scale variation in a parameter ν , which separates the phase space for collinear and soft particles along a hyperbola in the (p_-, p_+) plane. In contrast to the standard RG, there is no running coupling involved in the ν evolution, since the different contributions live at the same virtuality.

As an important application, consider the Higgs-boson production cross section in gluon fusion at the LHC, defined with a jet veto stating that no jet in the final state has transverse momentum above a threshold p_T^{veto} , which can be factorized in the form [78, 79]

$$\begin{aligned} \sigma(p_T^{\text{veto}}) = & H(m_H, \mu) \left(\frac{\nu_B}{\nu_S} \right)^{-2F_{gg}(R, p_T^{\text{veto}}, \mu)} S_{gg}(R, p_T^{\text{veto}}, \mu, \frac{\nu_S}{p_T^{\text{veto}}}) \\ & \times \int_{\tau}^1 \frac{dz}{z} B_{g/P} \left(z, R, p_T^{\text{veto}}, \mu, \frac{\nu_B}{m_H} \right) B_{g/P} \left(\frac{\tau}{z}, R, p_T^{\text{veto}}, \mu, \frac{\nu_B}{m_H} \right), \end{aligned} \quad (16.23)$$

where $\tau = m_H^2/s$, and $\mu \sim p_T^{\text{veto}}$ is a common factorization scale. The beam functions $B_{g/P}$, the soft function S_{gg} and the exponent F_{gg} all depend on the jet radius R as well as the jet clustering algorithm. The scale dependence of the hard function H is controlled by standard RG evolution in SCET. The beam functions can be factorized further into calculable collinear kernels convoluted with parton distribution functions. In addition to the renormalization scale μ , the beam and soft functions depend on two rapidity scales $\nu_B \sim m_H$ and $\nu_S \sim p_T^{\text{veto}}$, respectively. In [78] the default values $\nu_B = m_H$ and $\nu_S = p_T^{\text{veto}}$ are used for these scales, and the soft function S_{gg} is absorbed into the beam functions. In [79] the exponent F_{gg} is called $-\gamma_{\nu}^g/2$. The second factor on the right-hand side of the factorization formula (16.23), which resums large rapidity logarithms, implies that the logarithm of the jet-veto cross section contains a single large logarithm $\ln \sigma = -2F_{gg}(R, p_T^{\text{veto}}, \mu) \ln(m_H/p_T^{\text{veto}}) + \dots$ not contained in the hard function. Its coefficient can be calculated in fixed-order perturbation theory.

SCET_{II} also plays an important role in the study of factorization for a variety of exclusive B meson decays, such as $\bar{B} \rightarrow \pi \ell \bar{\nu}$, $\bar{B} \rightarrow K^* \gamma$ and $\bar{B} \rightarrow \pi \pi$, for which the virtualities of energetic (collinear) final-state particles are of order Λ_{QCD} , which is also the scale for the soft light degrees of freedom contained in the initial-state B meson.

16.3.7 Applications

Most of the applications of SCET are either in flavor physics, where the decay of a heavy B meson can give rise to energetic light partons, or in collider physics, where the presence of jets naturally leads to collimated sets of energetic particles. For some of these applications alternative approaches existed before the invention of SCET, but the effective theory has opened up alternative ways to understand the physics of these processes. For many examples, however, SCET has allowed new insights and new applications.

The investigation of heavy-to-light form factors has been instrumental for understanding factorization in exclusive semileptonic B decays [80]. SCET has also provided a field-theoretic basis for

the QCD factorization approach to exclusive, non-leptonic decays of B mesons [81]. Using SCET methods, proofs of factorization were derived for the color-allowed decay $\bar{B}^0 \rightarrow D^+\pi^-$ [82], the color-suppressed decay $\bar{B}^0 \rightarrow D^0\pi^0$ [83], and the radiative decay $\bar{B} \rightarrow K^*\gamma$ [84]. In recent years, the factorization approach has been extended to include QED effects [85,86]. Further examples are factorization theorems and the resummation of endpoint logarithms for quarkonia production [87], the resummation of large logarithmic terms for the thrust [88] and jet broadening [89] distributions in e^+e^- annihilation beyond NLL order, the development of new factorizable observables to veto extra jets [67,90], all-order factorization theorems for processes containing electroweak Sudakov logarithms [91], and the resummation of threshold (soft gluon) logarithms in momentum space for several important processes at hadron colliders [64,92,93]. There has also been a lot of activity describing p_T -based resummation at hadron colliders. Prominent examples are the transverse-momentum distributions of electroweak bosons [76,77,94], which give access to the p_T -dependent parton distribution functions. Higher-order resummations of event shape distributions [88], combined with two-loop fixed-order expressions [95,96] and including nonperturbative effects by means of a shape function, have been used for an accurate determination of the strong coupling constant α_s [97,98]. A method based on N -jettiness ($\mathcal{T}_N$) slicing [99,100] has provided a new way of performing the IR subtractions needed for precise fixed-order calculations, using the fact that the singular dependence on $\mathcal{T}_N$ can be calculated using SCET at NNLO. This has been used to compute various processes with final states containing up to one hard, colored particle [101–105]. Calculations of the leading power corrections in $\mathcal{T}_0/Q$ [106,107] have helped to improve the numerical stability of the method. The N -jettiness slicing method has also been used in the combination of higher-order resummation with parton showers [108–110]. Finally, SCET has given new insights into the jet substructure methods (see [111] for a review).

A novel type of SCET factorization theorem is needed to describe non-global observables, in which energetic radiation is limited to certain regions of phase space, e.g. inside jets. Jet events play a prominent role in collider physics, because they closely mirror the underlying short-distance dynamics. In the theoretical description of such observables one encounters so-called “non-global” logarithms (NGLs) [112,113], whose resummation cannot be performed using standard techniques. Using SCET, this problem has been reduced to the solution of a RG evolution equation derived from a novel type of factorization theorem, in which the hard and soft functions are operators in color space and in the (infinite) space of parton multiplicities [72,74] (see also [73,114]). In e^+e^- collisions, solving the RGE in the large- N_c limit using parton-shower techniques [115] gives a result that agrees with the solution of the BSM integral equation for the NGLs [116]. The first application of this formalism was to the light jet mass distribution [117], and the extension of the resummation to NLL accuracy has been accomplished in [118,119]. Non-global observables at hadron colliders are yet more intricate, since Glauber-gluon exchange between the colliding partons lead to a new type of double logarithmic corrections in high orders of perturbation theory, the so-called superleading logarithms (SLLs) [120,121]. Based on the corresponding SCET factorization theorem, the all-order resummation of SLLs has been accomplished for arbitrary $pp \rightarrow n$ jet processes [122].

Recent years have seen a strong effort to push the applications of SCET factorization and resummation techniques to next-to-leading power (NLP) in the expansion in λ . The subleading SCET Lagrangian [60,123,124] and current operators arising in B -meson decays and their anomalous dimensions [59,60,125–127] have been studied a long time ago. For important collider processes, such as Drell-Yan or Higgs production, the general sets of operators have been identified [128–130], and several of their anomalous dimensions have been calculated [130,131]. Resummed results at subleading power have been presented for event shapes [132] and the Drell-Yan process [133]. More applications of SCET can be found in Section 6.4 of the review article [134].

16.4 Open issues and perspectives

HQET has successfully passed many experimental tests, and there are not many open questions that still need to be addressed. One concept that has not been derived from first principles is the notion of quark-hadron duality, which underlies the application of HQET to the description of inclusive decays of B mesons. The validity of global duality (at energies even lower than those relevant in B decays) has been tested experimentally using high-precision data on semileptonic B decays and on hadronic τ decays. However, assigning a theoretical uncertainty due to possible duality violations remains a difficult task. Another known issue is that the measured values of the CKM elements $|V_{ub}|$ and $|V_{cb}|$ extracted from exclusive or inclusive decays of B mesons differ from each other by several standard deviations (see Section 75). This measurement relies on the heavy-quark limit, and the uncertainty quoted includes a theoretical estimate of the effect of power corrections arising from the finite b -quark mass. It remains an open question whether the discrepancy is due to underestimated theoretical or experimental uncertainties.

SCET, on the other hand, is still an active field of research, and new results are being obtained regularly. In particular, it has been realized that at NLP SCET factorization theorems generically suffer from endpoint-divergent convolution integrals [135–140]. To remove these divergences, one uses exact D -dimensional refactorization conditions, which relate different terms in the NLP factorization theorem in the singular limits, to rearrange these terms in such a way that the endpoint divergences are systematically removed. Successful applications of this technique include the NLP factorization theorem for the b -quark induced contribution to the diphoton decay amplitude of the Higgs boson [139, 141] and the $gg \rightarrow H$ production amplitude [142], the “off-diagonal gluon thrust” in e^+e^- collisions [143], and the NLP factorization theorems for some exclusive [144, 145] and inclusive [146] B -meson decays. On the other hand, for some other processes, such as electron-muon backward scattering [147], the factorization of endpoint singularities is still a subject of ongoing research.

Another active field concerns the further study and applications of Glauber gluons and quarks in SCET and their relation to the BFKL equation familiar from small- x physics [114, 148]. The theoretical formalism developed in [70, 71] sets the basis for a solid understanding of the impact of Glauber exchanges on factorization theorems. Moreover, using the rapidity renormalization group one can derive the BFKL equation in the presence of a single Glauber exchange describing forward scattering [70, 148] and for small x resummation in deep inelastic scattering [149]. Glauber gluons play an important role in SCET-based analysis of jet propagation in dense QCD media [150–152], which gives rise to the jet-quenching phenomenon in heavy-ion collisions. They are also a crucial ingredient in studies of violations of collinear factorization at hadron colliders [69, 153–157].

We close this short review by mentioning a particularly nice application combining the methods of heavy-particle EFTs such as HQET and non-relativistic QCD with SCET in the context of describing the interactions of heavy dark matter (with mass $M \gg v$) with SM particles. In [158] it was realized that the interactions of heavy, weakly interacting massive particles (WIMPs) with nuclear targets can be described in a model-independent way using heavy-particle EFTs. The WIMPs are charged under $SU(2)_L$ and can interact with electroweak gauge bosons and the Higgs boson. The WIMP EFT was later extended by describing the produced, highly energetic electroweak gauge bosons in terms of soft or collinear fields in SCET [159–161]. This allows one to systematically separate all relevant mass scales, resum electroweak Sudakov logarithms and disentangle the so-called Sommerfeld enhancement from the short-distance hard annihilation process.

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