Figure 10.5 shows these and other quantities sometimes used to describe multiple Coulomb scattering. They are

$$\psi^{\,\rm rms}_{\rm plane} = \frac{1}{\sqrt{3}}\,\theta^{\,\rm rms}_{\rm plane} = \frac{1}{\sqrt{3}}\,\theta_0 \ , \tag{10.11}$$

$$y^{\,\rm rms}_{\rm plane} = \frac{1}{\sqrt{3}}\,x\,\theta^{\,\rm rms}_{\rm plane} = \frac{1}{\sqrt{3}}\,x\,\theta_0 \ , \tag{10.12}$$

$$s^{\,\rm rms}_{\rm plane} = \frac{1}{4\sqrt{3}}\,x\,\theta^{\,\rm rms}_{\rm plane} = \frac{1}{4\sqrt{3}}\,x\,\theta_0 \ . \tag{10.13}$$

All the quantitative estimates in this section apply only in the limit of small $\theta^{\,\rm rms}_{\rm plane}$ and in the absence of large-angle scatters. The random variables s, ψ, y, and θ in a given plane are distributed in a correlated fashion (see Sec. 19.1 of this *Review* for the definition of the correlation coefficient). Obviously, $y \approx x\psi$. In addition, y and θ have the correlation coefficient $\rho_{y\theta} = \sqrt{3}/2 \approx 0.87$. For Monte Carlo generation of a joint $(y_{\rm \,plane}, \theta_{\rm plane})$ distribution, or for other calculations, it may be most convenient to work with independent Gaussian random variables (z_1, z_2) with mean zero and variance one, and then set

$$\begin{aligned} y_{\rm plane} =& z_1\, x\, \theta_0 (1-\rho^2_{y\theta})^{1/2}/\sqrt{3} + z_2\, \rho_{y\theta} x\, \theta_0/\sqrt{3} \\ =& z_1\, x\, \theta_0/\sqrt{12} + z_2\, x\, \theta_0/2 \ ; \end{aligned} \tag{10.14}$$

$$\theta_{\rm plane} = z_2\, \theta_0 \ . \tag{10.15}$$

Note that the second term for $y_{\rm \,plane}$ equals $x\,\theta_{\rm plane}/2$ and represents the displacement that would have occurred had the deflection $\theta_{\rm plane}$ all occurred at the single point $x/2$.

For heavy ions the multiple Coulomb scattering has been measured and compared with various theoretical distributions [26].

10.7. Radiation length and associated quantities

In dealing with electrons and photons at high energies, it is convenient to measure the thickness of the material in units of the radiation length X_0. This is the mean distance over which a high-energy electron loses all but $1/e$ of its energy by bremsstrahlung, and is the appropriate scale length for describing high-energy electromagnetic cascades. X_0 has been calculated and tabulated by Y.S. Tsai [27]:

$$\frac{1}{X_0} = 4\alpha r_e^2 \frac{N_A}{A}\Big\{Z^2\big[L_{\rm rad} - f(Z)\big] + Z\,L'_{\rm rad}\Big\} \tag{10.16}$$

$L_{\rm rad}$ and $L'_{\rm rad}$ are given in Table 10.2. The function $f(Z)$ is an infinite sum, but for elements up to uranium can be represented to 4-place accuracy by

$$\begin{aligned} f(Z) = a^2\big[&(1+a^2)^{-1} + 0.20206 \\ &-0.0369\,a^2 + 0.0083\,a^4 - 0.002\,a^6\big] \ , \end{aligned} \tag{10.17}$$

where $a = \alpha Z$ [28].

Table 10.2: Tsai's $L_{\rm rad}$ and $L'_{\rm rad}$, for use in calculating the radiation length in an element using Eq. (10.16).

Element	Z	$L_{\rm rad}$	$L'_{\rm rad}$
H	1	5.31	6.144
He	2	4.79	5.621
Li	3	4.74	5.805
Be	4	4.71	5.924
Others	>4	$\ln(184.15\,Z^{-1/3})$	$\ln(1194\,Z^{-2/3})$

Although it is easy to use Eq. (10.16) to calculate X_0, the functional dependence on Z is somewhat hidden. Dahl provides a compact fit to the data [29]:

$$X_0 = \frac{716.4\ {\rm g\ cm}^{-2}\,A}{Z(Z+1)\ln(287/\sqrt{Z})} \tag{10.18}$$

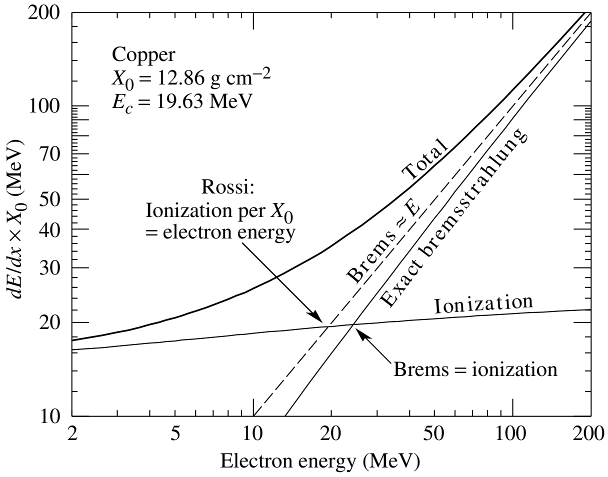

Figure 10.6: Two definitions of the critical energy E_c.

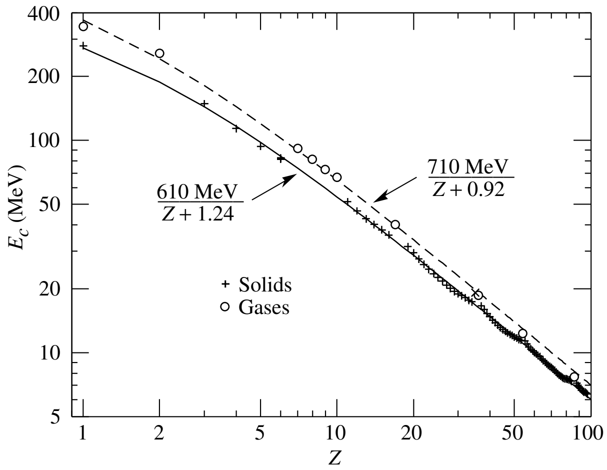

Figure 10.7: Critical energy for the chemical elements, using Rossi's definition [1]. The fits shown are for solids and liquids (solid line) and gases (dashed line). The rms deviation is 2.2% for the solids and 4.0% for the gases.

Results obtained with this formula agree with Tsai's values to better than 2.5% for all elements except helium, where the result is about 5% low.

The radiation length in a mixture or compound may be approximated by

$$1/X_0 = \sum w_j/X_j \ , \tag{10.19}$$

where w_j and X_j are the fraction by weight and the radiation length for the jth element.

An electron loses energy by bremsstrahlung at a rate nearly proportional to its energy, while the ionization loss rate varies only logarithmically with the electron energy. The *critical energy* E_c is sometimes defined as the energy at which the two loss rates are equal [30]. Berger and Seltzer [30] also give the approximation $E_c = (800\ {\rm MeV})/(Z+1.2)$. This formula has been widely quoted, and has been given in previous editions of this *Review* [17]. Among alternate definitions is that of Rossi [1], who defines the critical energy as the energy at which the ionization loss per radiation length is equal to the electron energy. Equivalently, it is the same as the first definition with the approximation $|dE/dx|_{\rm brems} \approx E/X_0$. These definitions are illustrated in the case of copper in Fig. 10.6 [31].

The accuracy of approximate forms for E_c has been limited by the failure to distinguish between gases and solid or liquids, where there is a substantial difference in ionization at the relevant energy because of the density effect. We distinguish these two cases in Fig. 10.7. Fits