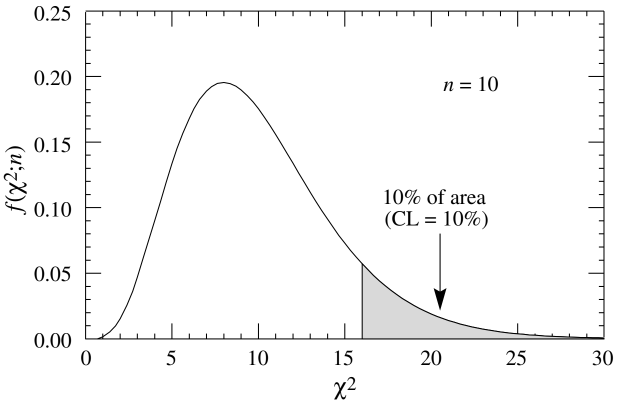

Figure 16.2: Illustration of the confidence level integral given in Eq. (16.22). This particlar example is for $n = 10$, where the area above 15.99 is 0.1.

Since the mean of the χ^2 distribution is equal to n, one expects in a "reasonable" experiment to obtain $\chi^2 \approx n$. While caution is necessary because of the width and skewness of the distribution, the "reduced χ^2" $\equiv \chi^2/n$ is a sometimes useful quantity. Figure 16.3 shows χ^2/n for useful CL's as a function of n.

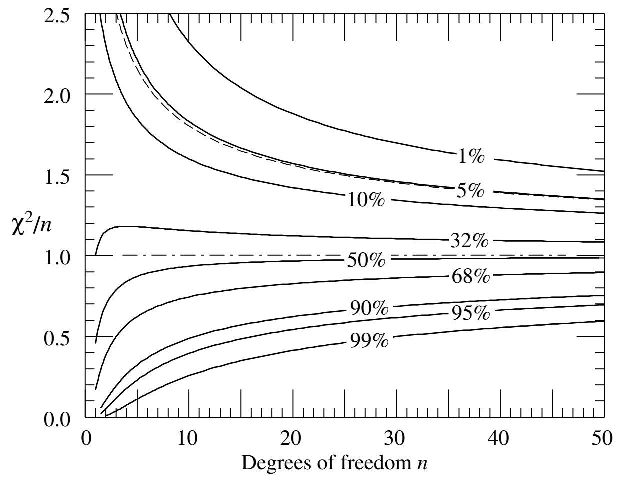

Figure 16.3: Confidence levels as a function of the "reduced χ^2" $\equiv \chi^2/n$ and the number of degrees of freedom n. Curves are labeled by the probability that a measurement will give a value of χ^2/n greater than that given on the y axis; *e.g.*, for $n = 10$, a value $\chi^2/n \gtrsim 1.8$ can be expected 5% of the time.

For large n, the CL is approximately given by [1,7]

$$\mathrm{CL}(\chi^2) \approx \frac{1}{\sqrt{2\pi}} \int_y^\infty e^{-x^2/2}\, dx \ , \qquad (16.23)$$

where $y = \sqrt{2\chi^2} - \sqrt{2n-1}$. This approximation was used to draw the dashed curves in Fig. 16.1 (for $n = 20$) and Fig. 16.3 (for CL = 5%). Since all the functions and their inverses are now readily available in standard mathematical libraries (such as IMSL, used to generate these figures, and personal computer spreadsheets, such as Microsoft®Excel [8]), the approximation (and even figures and tables) are seldom needed.

16.3.5. *Student's t distribution*: Suppose that x and $x_1, \ldots, x_n$ are independent and Gaussian distributed with mean 0 and variance 1. We then define

$$z = \sum_1^n x_i^2 \ , \quad \text{and} \quad t = \frac{x}{\sqrt{z/n}} \ . \qquad (16.24)$$

The variable z thus belongs to a $\chi^2(n)$ distribution. Then t is distributed according to a Student's t distribution with n degrees of freedom, $f(t;n)$, given in Table 16.1.

The Student's t distribution resembles a Gaussian distribution with wide tails. As $n \to \infty$, the distribution approaches a Gaussian. If $n = 1$, the distribution is a *Cauchy* or *Breit-Wigner* distribution. The mean is finite only for $n > 1$ and the variance is finite only for $n > 2$, so for $n = 1$ or $n = 2$, t does not obey the central limit theorem.

As an example, consider the *sample mean* $\overline{x} = \sum x_i/n$ and the *sample variance* $s^2 = \sum(x_i - \overline{x})^2/(n-1)$ for normally distributed random variables x_i with unknown mean μ and variance σ^2. The sample mean has a Gaussian distribution with a variance σ^2/n, so the variable $(\overline{x} - \mu)/\sqrt{\sigma^2/n}$ is normal with mean 0 and variance 1. Similarly, $(n-1)\, s^2/\sigma^2$ is independent of this and is χ^2 distributed with $n - 1$ degrees of freedom. The ratio

$$t = \frac{(\overline{x} - \mu)/\sqrt{\sigma^2/n}}{\sqrt{(n-1)\, s^2/\sigma^2\, (n-1)}} = \frac{\overline{x} - \mu}{\sqrt{s^2/n}} \qquad (16.25)$$

is distributed as $f(t;\, n-1)$. The unknown true variance σ^2 cancels, and t can be used to test the probability that the true mean is some particular value μ.

In Table 16.1, n in $f(t;n)$ is not required to be an integer. A Student's t distribution with nonintegral $n > 0$ is useful in certain applications.

16.3.6. *Gamma distribution*: For a process that generates events as a function of x (*e.g.*, space or time) according to a Poisson distribution, the distance in x from an arbitrary starting point (which may be some particular event) to the k^{th} event belongs to a *gamma* distribution, $f(x;\, \lambda,\, k)$. The Poisson parameter μ is λ per unit x. The special case $k = 1$ (*i.e.*, $f(x;\lambda,1) = \lambda e^{-\lambda x}$) is called the *exponential* distribution. A sum of k' exponential random variables x_i is distributed as $f(\sum x_i;\, \lambda, k')$.

The parameter k is not required to be an integer. For $\lambda = 1/2$ and $k = n/2$, the gamma distribution reduces to the $\chi^2(n)$ distribution.

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8. Microsoft® is a registered trademark of Microsoft corporation.