Table 17.1: Area of the tails ε outside $\pm\delta$ from the mean of a Gaussian distribution.

ε (%)	δ	ε (%)	δ
31.73	1σ	20	1.28σ
4.55	2σ	10	1.64σ
0.27	3σ	5	1.96σ
6.3×10^{-3}	4σ	1	2.58σ
5.7×10^{-5}	5σ	0.1	3.29σ
2.0×10^{-7}	6σ	0.01	3.89σ

in the fit as χ^2 decreases. In an attempt to reduce possible confusion in this discussion, we will use the ε notation (which corresponds to notation used in hypothesis testing [4]) when discussing confidence intervals and CL notation when discussing goodness-of-fit. Elsewhere in this *Review*, where the confusion between fit confidence level and interval (usually an upper or lower limit) confidence level does not arise, we follow the common practice of using "CL" to refer to the confidence level of the interval. This CL is understood to represent $1-\varepsilon$.

If the variance σ^2 of the estimator is not known, but must be estimated from the data, then we need to incorporate the error in $\widehat{\sigma}$ into our confidence interval using Student's t distribution. If we have N data points with which we estimate k parameters, the Gaussian approximation is adequate for $N-k \gg 1$. Otherwise replace δ by a factor $T\widehat{\sigma}$, T being defined by

$$1-\varepsilon = \int_{-T}^{T} f(t; N-k)\; dt \ , \tag{17.30}$$

where f for the Student's t-distribution is defined in Table 16.1. T is tabulated in Ref. 13 and in Table 17.2.

Table 17.2: t limits containing $1-\varepsilon$ of the area of Student's t-distribution $f(t; N-k)$.

	ε (%)					
$N-k$	31.73	10.00	5.00	4.55	1.00	0.27
1	1.84	6.31	12.71	13.97	63.66	235.8
2	1.32	2.92	4.30	4.53	9.92	19.21
3	1.20	2.35	3.18	3.31	5.84	9.22
4	1.14	2.13	2.78	2.87	4.60	6.62
5	1.11	2.01	2.57	2.65	4.03	5.51
10	1.05	1.81	2.23	2.28	3.17	3.96
20	1.03	1.72	2.09	2.13	2.85	3.42
∞	1.00	1.64	1.96	2.00	2.58	3.00

For multivariate $\boldsymbol{\alpha}$ we must consider pairwise correlations. Assuming a multivariate Gaussian, Eq. (16.20), and subsequent discussion the standard error ellipse for the pair $(\widehat{\alpha}_m, \widehat{\alpha}_n)$ may be drawn as in Fig. 17.3.

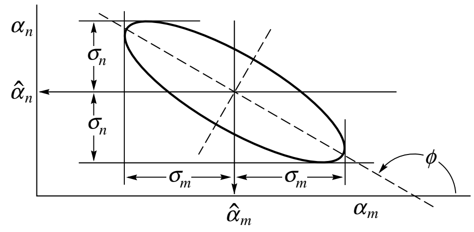

Figure 17.3: Standard error ellipse for the estimators $\widehat{\alpha}_m$ and $\widehat{\alpha}_n$. In this case the correlation is negative.

The minimum χ^2 or maximum likelihood solution is at $(\widehat{\alpha}_m, \widehat{\alpha}_n)$. The standard errors σ_m and σ_n are defined as shown, where the ellipse is at a constant value of $\chi^2 = \chi^2_{\min} + 1$ or $\ln\mathscr{L} = \ln\mathscr{L}_{\max} - 1/2$. The angle of the major axis of the ellipse is given by

$$\tan 2\phi = \frac{2\rho_{mn}\,\sigma_m\,\sigma_n}{\sigma_m^2 - \sigma_n^2} \ . \tag{17.31}$$

For non-Gaussian or nonlinear cases, one may construct an analogous contour from the same χ^2 or $\ln\mathscr{L}$ relations. Any other parameters $\widehat{\alpha}_\ell, \ell \neq m, n$ must be allowed freely to find their optimum values for every trial point.

For any unbiased procedure (*e.g.*, least squares or maximum likelihood) being used to estimate k parameters α_i, $i = 1, \ldots, k$, the probability $1-\varepsilon$ that the true values of all k lie within the s-standard deviation ellipsoid may be found from Fig. 16.1. Read the ordinate as ε; the correct value of ε occurs on the $n = k$ curve at $\chi^2 = s^2$. For example, for $k = 2$, the probability that the true values of α_1 and α_2 simultaneously lie within the one-standard-deviation error ellipse $(s = 1)$, centered on $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$, is 39%. This probability only assumes Gaussian errors, unbiased estimators, and that the model describing the data in terms of the α_i is correct.

17.5.2. ***Poisson processes—upper limits***:

Because the outcome of a Poisson process is an integral number of events, n_0, it is usually not possible to set confidence intervals for the true Poisson parameter μ at a certain exact ε. For large n_0 an approximate interval can be set using the Gaussian approximation, in our section on Probability, Sec. 16.3.2, and the techniques of Sec. 17.5.1.

For small n_0 we can define an upper limit N for μ as being that value of μ such that it would be at least $1-\varepsilon$ (*e.g.*, 90% or 95%) probable that a random observation of n would then lie above the observed n_0. Thus

$$1-\varepsilon = \sum_{n=n_0+1}^{\infty} f(n;N) \ ; \qquad \varepsilon = \sum_{n=0}^{n_0} f(n;N) \ . \tag{17.32}$$

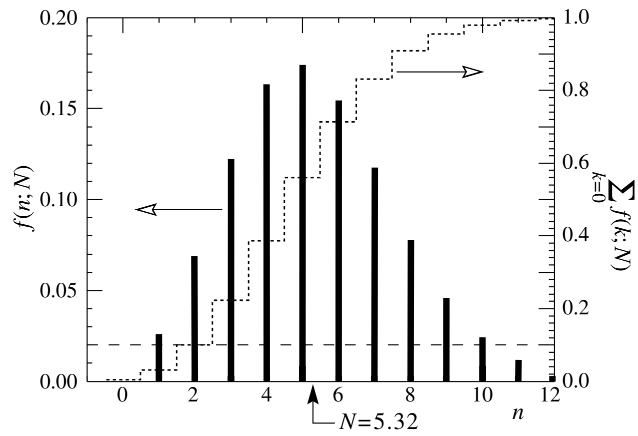

Figure 17.4: Illustration of Eq. (17.32) Poisson probabilities for an assumed mean of N. With an observed count $n_0 = 2$, $N = 5.32$ as shown gives summed probability $\varepsilon = 10\%$. The dotted summed probability curve (scale on right) has been displaced by -0.5 for clarity.

Fig. 17.4 illustrates the case with $n_0 = 2$ and $1-\varepsilon = 90\%$, for which it may be shown that $N = 5.32$. For any given n_0 and desired ε we can obtain N from the χ^2 Confidence Level figure because of a relation between the Poisson and the χ^2 distributions: read the ordinate as ε, find χ^2 on the curve for $n = 2(n_0+1)$; then $N = \chi^2/2$. Some useful values are given in Table 17.3.

The meaning of these upper limits is that, for a given true μ, the probability is *at least* $1-\varepsilon$ that one will observe n_0 which will result in N which is $\geq \mu$. The probability for that to occur may be higher than $1-\varepsilon$; for example, if $\mu \leq 2.30$ a "90%" upper limit will actually exceed μ 100% of the time. Note from Eq. (17.32) that for $n_0 = 0$, $N = -\ln\varepsilon$.