23. KINEMATICS

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Throughout this section units are used in which $\hbar = c = 1$. The following conversions are useful: $\hbar c = 197.3$ MeV fermi, $(\hbar c)^2 = 0.3894$ (GeV)^2 mb.

23.1. Lorentz transformations

The energy E and 3-momentum $\boldsymbol{p}$ of a particle of mass m form a 4-vector $p = (E, \boldsymbol{p})$ whose square $p^2 \equiv E^2 - |\boldsymbol{p}|^2 = m^2$. The velocity of the particle is $\boldsymbol{\beta} = \boldsymbol{p}/E$. The energy and momentum $(E^*, \boldsymbol{p}^*)$ viewed from a frame moving with velocity $\boldsymbol{\beta}_f$ are given by

$$\begin{pmatrix} E^* \\ p_\parallel^* \end{pmatrix} = \begin{pmatrix} \gamma_f & -\gamma_f\beta_f \\ -\gamma_f\beta_f & \gamma_f \end{pmatrix} \begin{pmatrix} E \\ p_\parallel \end{pmatrix} , \quad p_T^* = p_T , \tag{23.1}$$

where $\gamma_f = (1 - \beta_f^2)^{-1/2}$ and p_T $(p_\parallel)$ are the components of $\boldsymbol{p}$ perpendicular (parallel) to $\boldsymbol{\beta}_f$. The scalar product of two 4-vectors $p_1 \cdot p_2 = E_1 E_2 - \boldsymbol{p}_1 \cdot \boldsymbol{p}_2$ is invariant (frame independent).

In the collision of two particles of masses m_1 and m_2 the total center-of-mass energy can be expressed in the Lorentz-invariant form

$$\begin{aligned} E_{\rm cm} &= \Big[(E_1 + E_2)^2 - (\boldsymbol{p}_1 + \boldsymbol{p}_2)^2\Big]^{1/2} , \\ &= \Big[m_1^2 + m_2^2 + 2E_1E_2(1 - \beta_1\beta_2\cos\theta)\Big]^{1/2} , \end{aligned} \tag{23.2}$$

where θ is the angle between the particles. In the frame where one particle (of mass m_2) is at rest (lab frame),

$$E_{\rm cm} = (m_1^2 + m_2^2 + 2E_{1\,\rm lab}\, m_2)^{1/2} . \tag{23.3}$$

The velocity in the lab of the center-of-mass frame is

$$\boldsymbol{\beta}_{\rm cm} = \boldsymbol{p}_{\rm lab}/(E_{1\,\rm lab} + m_2) , \tag{23.4}$$

and

$$\gamma_{\rm cm} = (E_{1\,\rm lab} + m_2)/E_{\rm cm} . \tag{23.5}$$

23.2. Center-of-mass energy and momentum

A beam of particles with mass m and momentum $p_{\rm beam}$ is incident on a fixed target consisting of particles with mass M. The energy of the beam particles $E_{\rm beam}$, the total center-of-mass energy $E_{\rm cm}$, and center of mass momentum of one of the particles $p_{\rm cm}$ are given by

$$E_{\rm beam} = \sqrt{p_{\rm beam}^2 + m^2} \tag{23.6}$$

$$E_{\rm cm} = \sqrt{m^2 + 2E_{\rm beam}M + M^2} \tag{23.7}$$

$$p_{\rm cm} = p_{\rm beam}\frac{M}{E_{\rm cm}} . \tag{23.8}$$

For example, if a 0.80 GeV/c kaon beam is incident on a proton target, the center of mass energy is 1.699 GeV and the center of mass momentum of either particle is 0.442 GeV/c. It is also useful to note that

$$E_{\rm cm}\, dE_{\rm cm} = M\, dE_{\rm beam} = M\,\beta_{\rm beam}\, dp_{\rm beam} . \tag{23.9}$$

23.3. Lorentz-invariant amplitudes

The invariant amplitude $-i\mathscr{M}$ for a scattering or decay process is determined in perturbation theory by a set of Feynman diagrams. The convention of Bjorken and Drell is used except that fermion spinors are normalized so that $u\overline{u} = 2m$. As an example, the S-matrix for $2 \to 2$ scattering is related to $\mathscr{M}$ by

$$\begin{aligned} \langle p_1' p_2' \,|S|\, p_1 p_2\rangle &= I - i(2\pi)^4\, \delta^4(p_1 + p_2 - p_1' - p_2') \\ &\times \frac{\mathscr{M}(p_1, p_2;\, p_1', p_2')}{(2E_1)^{1/2}\,(2E_2)^{1/2}\,(2E_1')^{1/2}\,(2E_2')^{1/2}} . \end{aligned} \tag{23.10}$$

The state normalization is such that

$$\langle p'|p\rangle = (2\pi)^3\delta^3(\boldsymbol{p} - \boldsymbol{p}') . \tag{23.11}$$

23.4. Particle decays

The partial decay rate of a particle of mass M into n bodies in its rest frame is given in terms of the Lorentz-invariant matrix element $\mathscr{M}$ by

$$d\Gamma = \frac{(2\pi)^4}{2M}\, |\mathscr{M}|^2\, d\Phi_n\, (P;\, p_1, \ldots, p_n), \tag{23.12}$$

where $d\Phi_n$ is an element of n-body phase space given by

$$d\Phi_n(P;\, p_1, \ldots, p_n) = \delta^4\, (P - \sum_{i=1}^n p_i)\, \prod_{i=1}^n \frac{d^3 p_i}{(2\pi)^3 2E_i} . \tag{23.13}$$

This phase space can be generated recursively, viz.

$$\begin{aligned} d\Phi_n(P;\, p_1, \ldots, p_n) &= d\Phi_j(q;\, p_1, \ldots, p_j) \\ &\times\; d\Phi_{n-j+1}\, (P;\, q, p_{i+1}, \ldots, p_n)(2\pi)^3 dq^2 , \end{aligned} \tag{23.14}$$

where $q^2 = (\sum_{i=1}^j E_i)^2 - \left|\sum_{i=1}^j \boldsymbol{p}_i\right|^2$. This form is particularly useful in the case where a particle decays into another particle which subsequently decays.

23.4.1. ***Survival probability***: If a particle of mass M has mean proper lifetime τ $(= 1/\Gamma)$ and has momentum $(E, \boldsymbol{p})$, then the probability that it lives for a time t_0 or greater before decaying is given by

$$P(t_0) = e^{-t_0\,\Gamma/\gamma} = e^{-Mt_0\,\Gamma/E} , \tag{23.15}$$

and the probability that it travels a distance x_0 or greater is

$$P(x_0) = e^{-Mx_0\,\Gamma/|\boldsymbol{p}|} . \tag{23.16}$$

23.4.2. ***Two-body decays***:

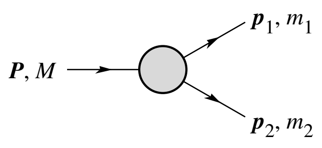

Figure 23.1: Variable definitions for two-body decays.

In the rest frame of a particle of mass M, decaying into 2 particles labeled 1 and 2,

$$E_1 = \frac{M^2 - m_2^2 + m_1^2}{2M} , \tag{23.17}$$

$$\begin{aligned} |\boldsymbol{p}_1| &= |\boldsymbol{p}_2| \\ &= \frac{\left[\left(M^2 - (m_1 + m_2)^2\right)\left(M^2 - (m_1 - m_2)^2\right)\right]^{1/2}}{2M} , \end{aligned} \tag{23.18}$$

and

$$d\Gamma = \frac{1}{32\pi^2}\, |\mathscr{M}|^2\, \frac{|\boldsymbol{p}_1|}{M^2}\, d\Omega , \tag{23.19}$$

where $d\Omega = d\phi_1 d(\cos\theta_1)$ is the solid angle of particle 1.