$\alpha_s(\mu)^{(n_f-1)}$ for $\mu = M_Q$. This matching is somewhat arbitrary; a different relation between $\Lambda^{(n_f)}$ and $\Lambda^{(n_f-1)}$ would result if $\mu = M_Q/2$ were used. In practice the differences between the new and old procedures are very small. $\Lambda^{(5)} = 200$ MeV corresponds to $\Lambda^{(4)} = 289$ MeV in the old scheme and $\Lambda^{(4)} = 280$ MeV in the one now adopted. Note that these differences between $\Lambda^{(5)}$ and $\Lambda^{(4)}$ are numerically very significant.

Data from deep-inelastic scattering are in a range of energy where the bottom quark is not readily excited and hence these experiments quote $\Lambda^{(4)}_{\overline{\rm MS}}$. Most data from PEP, PETRA, TRISTAN, and LEP quote a value of $\Lambda^{(5)}_{\overline{\rm MS}}$ since these data are in an energy range where the bottom quark is light compared to the available energy. We have converted it to $\Lambda^{(4)}_{\overline{\rm MS}}$ as required. A few measurements, including the lattice gauge theory values from the ψ system and from τ decay are at sufficiently low energy that $\Lambda^{(3)}_{\overline{\rm MS}}$ is appropriate.

We turn now to a discussion of renormalization-scheme dependence in QCD. Although necessarily rather technical, this discussion is vital to understanding how α_s (or Λ) values can be measured and compared. See the review by Duke and Roberts [6] for further details.

Consider a "typical" QCD cross section which, when calculated perturbatively, starts at $\mathcal{O}(\alpha_s)$:

$$\sigma = A_1\, \alpha_s + A_2\, \alpha_s^2 + \cdots\ . \tag{25.7}$$

The coefficients A_1, A_2 come from calculating the appropriate Feynman diagrams. In performing such calculations various divergences arise, and these must be regulated in a consistent way. This requires a particular renormalization scheme (RS). The most commonly used one is the modified minimal subtraction ($\overline{\rm MS}$) scheme [7]. This involves continuing momentum integrals from 4 to 4–2ϵ dimensions and then subtracting off the resulting $1/\epsilon$ poles and also $(\ln 4\pi - \gamma_E)$, which is another artifact of continuing the dimension. (Here γ_E is the Euler-Mascheroni constant.) To preserve the dimensionless nature of the coupling, a mass scale μ must also be introduced: $g \to \mu^\epsilon g$. The finite coefficients A_i thus obtained depend implicitly on the renormalization convention used and explicitly on the scale μ.

The first two coefficients (β_0, β_1) in Eq. (25.4) are independent of the choice of RS's. In contrast, the coefficients of terms proportional to α_s^n for $n > 3$ *are* RS-dependent. The form given above for β_2 is in the $\overline{\rm MS}$ scheme. It has become conventional to use the $\overline{\rm MS}$ scheme for calculating QCD cross sections beyond leading order.

The fundamental theorem of RS dependence is straightforward. Physical quantities, in particular the cross section, calculated to all orders in perturbation theory, do not depend on the RS. It follows that a truncated series *does* exhibit RS dependence. In practice, QCD cross sections are known to leading or to next-to-leading order or, in a very few cases, to next-to-next-to-leading order, and it is only the latter two cases, which have reduced RS dependence, that are useful for precision tests. At second order the RS dependence is completely given by one condition which can be taken to be the value of the renormalization scale μ. One therefore has to address the question of what is the "best" choice for μ. There is no definite answer to this question—higher-order corrections do not "fix" the scale, rather they render the theoretical predictions less sensitive to its variation.

One could imagine that choosing a scale μ characteristic of the typical energy scale (E) in the process would be most appropriate. In general a poor choice of scale generates terms of order $\ln(E/\mu)$ in the A_i's. More byzantine choices are the scale for which the next-to-leading-order correction vanishes ("Fastest Apparent Convergence [8]"), the scale for which the next-to-leading-order prediction is stationary [9] (*i.e.* the value of μ where $d\sigma/d\mu = 0$), or that dictated by the effective charge scheme [10].

An important corollary is that if the higher-order corrections are naturally small, then the additional uncertainties introduced by the μ dependence are likely to be less than the experimental measurement errors. There are some processes, however, for which the choice of scheme (*i.e.* the value of μ) *can* influence the extracted value of $\Lambda_{\overline{\rm MS}}$. There is no resolution to this problem other than to try to calculate even more terms in the perturbation series.

In the cases where the higher-order corrections to a process are known and are large, some caution should be exercised when quoting the value of α_s. In what follows we will attempt to indicate the size of the theoretical uncertainties on the extracted value of α_s. There are two simple ways to determine this error. First, we can estimate it by comparing the value of $\alpha_s(\mu)$ obtained by fitting data using the QCD formula to highest known order in α_s, and then comparing it with the value obtained using the next-to-highest-order formula (μ is chosen as the typical energy scale in the process). The corresponding Λ's are then obtained by evolving $\alpha_s(\mu)$ to $\mu = m_Z$ using Eq. (25.4) to the same order in α_s as the fit, and then converting to $\Lambda^{(4)}$ using Eq. (25.6). Alternatively, we can vary the value of μ over a reasonable range, extracting a value of Λ for each choice of μ. This method is of its nature imprecise since "reasonable" involves a subjective judgement. In either case, if the perturbation series is well behaved, the resulting error on Λ will be small.

25.3. QCD in deep-inelastic scattering

The original and still one of the most powerful quantitative tests of perturbative QCD is the breaking of Bjorken scaling in deep-inelastic lepton-hadron scattering. In the leading-logarithm approximation the measured structure functions $F_i(x, Q^2)$ are related to the quark distribution functions $q_i(x, Q^2)$ according to the naive parton model by the formulae in Sec. 24 of this *Review*. (In that section, q_i is denoted by the notation f_q). In describing the way in which scaling is broken in QCD, it is convenient to define nonsinglet and singlet quark distributions:

$$F^{NS} = q_i - q_j \qquad\qquad F^S = \sum_i (q_i + \overline{q}_i)\ . \tag{25.8}$$

The nonsinglet structure functions have nonzero values of flavor quantum numbers such as isospin or baryon number. The variation with Q^2 of these is described by the so-called Altarelli-Parisi equations [11]:

$$Q^2\,\frac{\partial F^{NS}}{\partial Q^2} = \frac{\alpha_s(|Q|)}{2\pi} P^{qq} * F^{NS} \tag{25.9a}$$

$$Q^2\,\frac{\partial}{\partial Q^2}\begin{pmatrix} F^S \\ G \end{pmatrix} = \frac{\alpha_s(|Q|)}{2\pi}\begin{pmatrix} P^{qq} & 2n_f P^{qg} \\ P^{gq} & P^{gg} \end{pmatrix} * \begin{pmatrix} F^S \\ G \end{pmatrix} \tag{25.9b}$$

where $*$ denotes a convolution integral:

$$f * g = \int_x^1 \frac{dy}{y}\, f(y)\, g\left(\frac{x}{y}\right)\ . \tag{25.10}$$

The leading-order Altarelli-Parisi splitting functions are

$$P^{qq} = \frac{4}{3}\left[\frac{1+x^2}{(1-x)_+}\right] + 2\delta(1-x)\ , \tag{25.11a}$$

$$P^{qg} = \frac{1}{2}\left[x^2 + (1-x)^2\right]\ , \tag{25.11b}$$

$$P^{gq} = \frac{4}{3}\left[\frac{1+(1-x)^2}{x}\right]\ , \tag{25.11c}$$

$$P^{gg} = 6\left[\frac{1-x}{x} + x(1-x) + \frac{x}{(1-x)_+} + \frac{11}{12}\delta(1-x)\right] - \frac{n_f}{3}\delta(1-x)\ . \tag{25.11d}$$

Here the gluon distribution $G(x, Q^2)$ has been introduced and $1/(1-x)_+$ means

$$\int_0^1 dx \frac{f(x)}{(1-x)_+} = \int_0^1 dx\, \frac{f(x) - f(1)}{(1-x)}\ . \tag{25.12}$$

The precision of contemporary experimental data demands that higher-order corrections also be included [12]. The above results are for massless quarks. Algorithms exist for the inclusion of nonzero quark