28. THE CABIBBO-KOBAYASHI-MASKAWA MIXING MATRIX

Updated 1993 by F.J. Gilman, K. Kleinknecht, and B. Renk.

In the Standard Model with SU(2) × U(1) as the gauge group of electroweak interactions, both the quarks and leptons are assigned to be left-handed doublets and right-handed singlets. The quark mass eigenstates are not the same as the weak eigenstates, and the matrix relating these bases was defined for six quarks and given an explicit parametrization by Kobayashi and Maskawa [1] in 1973. It generalizes the four-quark case, where the matrix is parametrized by a single angle, the Cabibbo angle [2].

By convention, the three charge 2/3 quarks (u, c, and t) are unmixed, and all the mixing is expressed in terms of a 3×3 unitary matrix V operating on the charge $-1/3$ quarks (d, s, b):

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} . \tag{28.1}$$

The values of individual matrix elements can in principle all be determined from weak decays of the relevant quarks, or, in some cases, from deep inelastic neutrino scattering. Using the constraints discussed below together with unitarity, and assuming only three generations, the 90% confidence limits on the magnitude of the elements of the complete matrix are:

$$\begin{pmatrix} 0.9747 \text{ to } 0.9759 & 0.218 \text{ to } 0.224 & 0.002 \text{ to } 0.005 \\ 0.218 \text{ to } 0.224 & 0.9738 \text{ to } 0.9752 & 0.032 \text{ to } 0.048 \\ 0.004 \text{ to } 0.015 & 0.030 \text{ to } 0.048 & 0.9988 \text{ to } 0.9995 \end{pmatrix} . \tag{28.2}$$

The ranges shown are for the individual matrix elements. The constraints of unitarity connect different elements, so choosing a specific value for one element restricts the range of the others.

There are several parametrizations of the Cabibbo-Kobayashi-Maskawa matrix. In view of the need for a "standard" parametrization in the literature, we advocate:

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix} \tag{28.3}$$

proposed by Chau and Keung [3]. The choice of rotation angles follows earlier work of Maiani [4], and the placement of the phase follows that of Wolfenstein [5]. The notation used is that of Harari and Leurer [6] who, along with Fritzsch and Plankl [7], proposed this parametrization as a particular case of a form generalizable to an arbitrary number of "generations." The general form was also put forward by Botella and Chau [8]. Here $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$, with i and j being "generation" labels, $\{i, j = 1, 2, 3\}$. In the limit $\theta_{23} = \theta_{13} = 0$ the third generation decouples, and the situation reduces to the usual Cabibbo mixing of the first two generations with θ_{12} identified with the Cabibbo angle [2]. The real angles θ_{12}, θ_{23}, θ_{13} can all be made to lie in the first quadrant by an appropriate redefinition of quark field phases. Then all s_{ij} and c_{ij} are positive, $|V_{us}| = s_{12}c_{13}$, $|V_{ub}| = s_{13}$, and $|V_{cb}| = s_{23}c_{13}$. As c_{13} is known to deviate from unity only in the fifth decimal place, $|V_{us}| = s_{12}$, $|V_{ub}| = s_{13}$, and $|V_{cb}| = s_{23}$ to an excellent approximation. The phase δ_{13} lies in the range $0 \le \delta_{13} < 2\pi$, with non-zero values generally breaking CP invariance for the weak interactions. The generalization to the n generation case contains $n(n-1)/2$ angles and $(n-1)(n-2)/2$ phases [6,7,8]. The range of matrix elements in Eq. (28.2) corresponds to 90% CL limits on the angles of $s_{12} = 0.218$ to $0.224, s_{23} = 0.032$ to 0.048, and $s_{13} = 0.002$ to 0.005.

Kobayashi and Maskawa [1] originally chose a parametrization involving the four angles, θ_1, θ_2, θ_3, δ:

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} c_1 & -s_1c_3 & -s_1s_3 \\ s_1c_2 & c_1c_2c_3-s_2s_3e^{i\delta} & c_1c_2s_3+s_2c_3e^{i\delta} \\ s_1s_2 & c_1s_2c_3+c_2s_3e^{i\delta} & c_1s_2s_3-c_2c_3e^{i\delta} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} , \tag{28.4}$$

where $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$ for $i = 1, 2, 3$. In the limit $\theta_2 = \theta_3 = 0$, this reduces to the usual Cabibbo mixing with θ_1 identified (up to a sign) with the Cabibbo angle [2]. Slightly different forms of the Kobayashi-Maskawa parametrization are found in the literature. The CKM matrix used in the 1982 *Review of Particle Properties* is obtained by letting $s_1 \to -s_1$ and $\delta \to \delta + \pi$ in the matrix given above. An alternative is to change Eq. (28.4) by $s_1 \to -s_1$ but leave δ unchanged. With this change in s_1, the angle θ_1 becomes the usual Cabibbo angle, with the "correct" sign (*i.e.* $d' = d\cos\theta_1 + s\sin\theta_1$) in the limit $\theta_2 = \theta_3 = 0$. The angles θ_1, θ_2, θ_3 can, as before, all be taken to lie in the first quadrant by adjusting quark field phases. Since all these parametrizations are referred to as "the" Kobayashi-Maskawa form, some care about which one is being used is needed when the quadrant in which δ lies is under discussion.

Other parametrizations, mentioned above, are due to Maiani [4] and to Wolfenstein [5]. The latter emphasizes the relative sizes of the matrix elements by expressing them in powers of the Cabibbo angle. Still other parametrizations [9] have come into the literature in connection with attempts to define "maximal CP violation". No physics can depend on which of the above parametrizations (or any other) is used as long as a single one is used consistently and care is taken to be sure that no other choice of phases is in conflict.

Our present knowledge of the matrix elements comes from the following sources:

(1) Nuclear beta decay, when compared to muon decay, gives [10–13]

$$|V_{ud}| = 0.9744 \pm 0.0010 . \tag{28.5}$$

This includes refinements in the analysis of the radiative corrections, especially the order $Z\alpha^2$ effects, which have brought the ft-values from low and high Z Fermi transitions into good agreement.

(2) Analysis of K_{e3} decays yields [14]

$$|V_{us}| = 0.2196 \pm 0.0023 . \tag{28.6}$$

(The notation K^+_{e3} refers to $K^+ \to \pi^0 e^+ \nu_e$ and K^0_{e3} refers to $K^0_L \to \pi^\pm e^\mp \nu_e$.) The isospin violation between K^+_{e3} and K^0_{e3} decays has been taken into account, bringing the values of $|V_{us}|$ extracted from these two decays into agreement at the 1% level of accuracy. The analysis of hyperon decay data has larger theoretical uncertainties because of first order SU(3) symmetry breaking effects in the axial-vector couplings, but due account of symmetry breaking [15] applied to the WA2 data [16] gives a corrected value [17] of 0.222 ± 0.003. We average these two results to obtain:

$$|V_{us}| = 0.2205 \pm 0.0018 . \tag{28.7}$$

(3) The magnitude of $|V_{cd}|$ may be deduced from neutrino and antineutrino production of charm off valence d quarks. The dimuon production cross sections of the CDHS group [18] yield $\overline{B}_c |V_{cd}|^2 = 0.41 \pm 0.07 \times 10^{-2}$, where $\overline{B}_c$ is the semileptonic branching fraction of the charmed hadrons produced. The corresponding value from a recent Tevatron experiment [19] is $\overline{B}_c |V_{cd}|^2 = 0.534^{+0.052}_{-0.078} \times 10^{-2}$. Averaging these two results gives $\overline{B}_c |V_{cd}|^2 = 0.47 \pm 0.05 \times 10^{-2}$. Supplementing this with measurements of the semileptonic branching fractions of charmed mesons [20], weighted by a production ratio of $D^0/D^+ = (60 \pm 10)/(40 \mp 10)$, to give $\overline{B}_c = 0.113 \pm 0.015$, yields

$$|V_{cd}| = 0.204 \pm 0.017 \tag{28.8}$$

(4) Values of $|V_{cs}|$ from neutrino production of charm are dependent on assumptions about the strange quark density in the parton-sea. The most conservative assumption, that the strange-quark sea does not exceed the value corresponding to an SU(3) symmetric sea, leads to a lower bound [18], $|V_{cs}| > 0.59$. It is more advantageous to proceed analogously to the method used for extracting $|V_{us}|$ from K_{e3} decay; namely, we compare the experimental value for the width of D_{e3} decay with the expression [21] that follows from the standard weak interaction amplitude:

$$\Gamma(D \to \overline{K} e^+ \nu_e) = |f^D_+(0)|^2 \, |V_{cs}|^2 \, (1.54 \times 10^{11} \text{ s}^{-1}) . \tag{28.9}$$