Here $f_+^D(q^2)$, with $q = p_D - p_K$, is the form factor relevant to D_{e3} decay; its variation has been taken into account with the parametrization $f_+^D(t)/f_+^D(0) = M^2/(M^2 - t)$ and $M = 2.1\ \mathrm{GeV}/c^2$, a form and mass consistent with Mark III and E691 measurements [22,23]. Combining data on branching ratios for $D_{\ell 3}$ decays from Mark III, E687, E691, and CLEO experiments [22–24] with accurate values [25] for τ_{D^+} and τ_{D^0}, gives the value $(0.762 \pm 0.055) \times 10^{11}\ \mathrm{s}^{-1}$ for $\Gamma(D \to \overline{K} e^+ \nu_e)$. Therefore

$$|f_+^D(0)|^2\ |V_{cs}|^2 = 0.495 \pm 0.036\ . \tag{28.10}$$

A very conservative assumption is that $|f_+^D(0)| < 1$, from which it follows that $|V_{cs}| > 0.62$. Calculations of the form factor either performed [26,27] directly at $q^2 = 0$, or done [28] at the maximum value of $q^2 = (m_D - m_K)^2$ and interpreted at $q^2 = 0$ using the measured q^2 dependence, yield $f_+^D(0) = 0.7 \pm 0.1$. It follows that

$$|V_{cs}| = 1.01 \pm 0.18\ . \tag{28.11}$$

The constraint of unitarity when there are only three generations gives a much tighter bound (see below).

(5) The ratio $|V_{ub}/V_{cb}|$ can be obtained from the semileptonic decay of B mesons produced on the $\Upsilon(4S)$ $b\bar{b}$ resonance by measuring the lepton energy spectrum above the endpoint of the $b \to c\ell\nu$ spectrum. There the $b \to u\ell\nu$ decay rate can be obtained by subtracting the background from nonresonant e^+e^- reactions. This continuum background is determined from auxiliary measurements off the $\Upsilon(4S)$. Both the CLEO [29] and ARGUS [30] collaborations have reported evidence for $b \to u$ transitions in semileptonic B decays. The interpretation of the result in terms of $|V_{ub}/V_{cb}|$ depends fairly strongly on the theoretical model used to generate the lepton energy spectrum, especially for $b \to u$ transitions [27,28,31]. Combining the experimental and theoretical uncertainties, we quote

$$|V_{ub}/V_{cb}| = 0.08 \pm 0.02\ . \tag{28.12}$$

(6) The magnitude of V_{cb} itself can be determined if the measured semileptonic bottom hadron partial width is assumed to be that of a b quark decaying through the usual $V - A$ interaction:

$$\Gamma(b \to c\ell\overline{\nu}_\ell) = \frac{\mathrm{B}(b \to c\ell\overline{\nu}_\ell)}{\tau_b} = \frac{G_F^2\, m_b^5}{192\pi^3} F(m_c/m_b)\ |V_{cb}|^2\ , \tag{28.13}$$

where τ_b is the b lifetime and $F(m_c/m_b)$ is a phase space factor that is approximately one-half. Most of the error on $|V_{cb}|$ derived from Eq. (28.13) is not from the experimental uncertainties, but in the theoretical uncertainties in choosing a value of m_b and in the use of the quark model to represent inclusively semileptonic decays which, at least for the B meson, are dominated by a few exclusive channels. Instead we use the nearly model-independent treatment in the heavy quark effective theory [32], where, in the case of $B \to D^*$ transitions, the decay rates at zero recoil are fixed by a normalization condition, with vanishing $1/m_q$ corrections [33]. From data of the ARGUS [34] and CLEO [35] experiments, we quote a value [36] derived from the decay of $\overline{B} \to D^* \ell \overline{\nu}_\ell$ of

$$|V_{cb}| = 0.040 \pm 0.005 \tag{28.14}$$

that is deduced using a B-lifetime of (1.49 ± 0.04) ps [37]. The central value and the error are now comparable to what is obtained from the inclusive semileptonic decays, but ultimately, with more data, exclusive semileptonic decays should provide the most accurate value of $|V_{cb}|$.

The results for three generations of quarks, from Eqs. 28.5, 28.7, 28.8, 28.11, 28.12, and 28.14 plus unitarity, are summarized in the matrix in Eq. (28.2). The ranges given there are different from those given in Eqs. (28.5)–(28.14) (because of the inclusion of unitarity), but are consistent with the one standard deviation errors on the input matrix elements.

The data do not preclude there being more than three generations. Moreover, the entries deduced from unitarity might be altered when the CKM matrix is expanded to accommodate more generations. Conversely, the known entries restrict the possible values of additional elements if the matrix is expanded to account for additional generations. For example, unitarity and the known elements of the first row require that any additional element in the first row have a magnitude $|V_{ub'}| < 0.07$. When there are more than three generations the allowed ranges (at 90% CL) of the matrix elements connecting the first three generations are

$$\begin{pmatrix} 0.9728 \text{ to } 0.9757 & 0.218 \text{ to } 0.224 & 0.002 \text{ to } 0.005 & \ldots \\ 0.180 \text{ to } 0.228 & 0.800 \text{ to } 0.975 & 0.032 \text{ to } 0.048 & \ldots \\ 0 \text{ to } 0.13 & 0 \text{ to } 0.56 & 0 \text{ to } 0.9995 & \ldots \\ \vdots & \vdots & \vdots & \end{pmatrix}, \tag{28.15}$$

where we have used unitarity (for the expanded matrix) and Eqs. 28.5, 28.7, 28.8, 28.11, 28.12, and 28.14.

Further information on the angles requires theoretical assumptions. For example, $B_d - \overline{B}_d$ mixing, if it originates from short distance contributions to ΔM_B dominated by box diagrams involving virtual t quarks, gives information on $V_{tb} V_{td}^*$ once hadronic matrix elements and the t quark mass are known. A similar comment holds for $V_{tb} V_{ts}^*$ and $B_s - \overline{B}_s$ mixing.

Direct and indirect information on the CKM matrix is neatly summarized in terms of the "unitarity triangle." The name arises since unitarity of the 3×3 CKM matrix applied to the first and third columns yields

$$V_{ud}\, V_{ub}^* + V_{cd}\, V_{cb}^* + V_{td} V_{tb}^* = 0\ . \tag{28.16}$$

The unitarity triangle is just a geometrical presentation of this equation in the complex plane [38]. We can always choose to orient the triangle so that $V_{cd}\, V_{cb}^*$ is horizontal; in the parametrization we have chosen, V_{cb} is real, and V_{cd} is real to a very good approximation in any case. Setting cosines of small angles to unity, Eq. (28.16) becomes

$$V_{ub}^* + V_{td} = s_{12}\, V_{cb}^*\ , \tag{28.17}$$

which is shown as the unitarity triangle in Fig. 28.1(a). Rescaling the triangle by a factor $[1/|s_{12}\, V_{cb}|]$, the coordinates of the vertices become

$$A\big(\mathrm{Re}(V_{ub})/|s_{12}\, V_{cb}|, -\mathrm{Im}(V_{ub})/|s_{12}\, V_{cb}|\big)\ ,\ B(1,0)\ ,\ C(0,0)\ . \tag{28.18}$$

In the approximation of the Wolfenstein parametrization [5], with matrix elements expressed in powers of the Cabibbo angle, $\lambda \sim s_{12}$:

$$\begin{aligned} V_{us} &\sim \lambda \\ V_{ub} &\sim \lambda^3 A(\rho - i\eta) \\ V_{cb} &\sim \lambda^2 A \\ V_{td} &\sim \lambda^3 A(1 - \rho - i\eta)\ , \end{aligned} \tag{28.19}$$

the coordinates of the vertex A of the unitarity triangle are simply (ρ, η), as shown in Fig. 28.1(b).

CP-violating processes will involve the phase in the CKM matrix, assuming that the observed CP violation is solely related to a nonzero value of this phase. This allows additional constraints to be imposed. More specifically, a necessary and sufficient condition for CP violation with three generations can be formulated in a parametrization-independent manner in terms of the non-vanishing of the determinant of the commutator of the mass matrices for the charge $2e/3$ and charge $-e/3$ quarks [39]. CP violating amplitudes or differences of rates all are proportional to the CKM factor in this quantity. This is the product of factors $s_{12} s_{13} s_{23} c_{12} c_{13}^2 c_{23} s_{\delta_{13}}$ in the parametrization adopted above, and is $s_1^2 s_2 s_3 c_1 c_2 c_3 s_\delta$ in that of Ref. 1. With the approximation of setting cosines to unity, this is just twice the area of the unitarity triangle. While hadronic matrix elements whose values are imprecisely known generally now enter, the constraints from CP violation in the neutral kaon system are tight enough to very much restrict the range of angles and the phase of