$$s^{\,\rm rms}_{\rm plane} = \frac{1}{4\sqrt{3}}\, x\, \theta^{\,\rm rms}_{\rm plane} = \frac{1}{4\sqrt{3}}\, x\, \theta_0 \ . \tag{10.13}$$

All the quantitative estimates in this section apply only in the limit of small $\theta^{\,\rm rms}_{\rm plane}$ and in the absence of large-angle scatters. The random variables s, ψ, y, and θ in a given plane are distributed in a correlated fashion (see Sec. 16.1 of this *Review* for the definition of the correlation coefficient). Obviously, $y \approx x\psi$. In addition, y and θ have the correlation coefficient $\rho_{y\theta} = \sqrt{3}/2 \approx 0.87$. For Monte Carlo generation of a joint $(y_{\,\rm plane}, \theta_{\rm plane})$ distribution, or for other calculations, it may be most convenient to work with independent Gaussian random variables (z_1, z_2) with mean zero and variance one, and then set

$$\begin{aligned} y_{\rm plane} =& z_1\, x\, \theta_0 (1 - \rho^2_{y\theta})^{1/2}/\sqrt{3} + z_2\, \rho_{y\theta} x\, \theta_0/\sqrt{3} \\ =& z_1\, x\, \theta_0/\sqrt{12} + z_2\, x\, \theta_0/2 \ ; \end{aligned} \tag{10.14}$$

$$\theta_{\rm plane} = z_2\, \theta_0 \ . \tag{10.15}$$

Note that the second term for $y_{\,\rm plane}$ equals $x\, \theta_{\rm plane}/2$ and represents the displacement that would have occurred had the deflection $\theta_{\rm plane}$ all occurred at the single point $x/2$.

For heavy ions the multiple Coulomb scattering has been measured and compared with various theoretical distributions [26].

10.7. Radiation length and associated quantities

In dealing with electrons and photons at high energies, it is convenient to measure the thickness of the material in units of the radiation length X_0. This is the mean distance over which a high-energy electron loses all but $1/e$ of its energy by bremsstrahlung, and is the appropriate scale length for describing high-energy electromagnetic cascades. X_0 has been calculated and tabulated by Y.S. Tsai [27]:

$$\frac{1}{X_0} = 4\alpha r_e^2 \frac{N_A}{A} \Big\{ Z^2 \big[L_{\rm rad} - f(Z) \big] + Z\, L'_{\rm rad} \Big\} \tag{10.16}$$

$L_{\rm rad}$ and $L'_{\rm rad}$ are given in Table 10.2. The function $f(Z)$ is an infinite sum, but for elements up to uranium can be represented to 4-place accuracy by

$$\begin{aligned} f(Z) = a^2 \big[& (1 + a^2)^{-1} + 0.20206 \\ & -0.0369\, a^2 + 0.0083\, a^4 - 0.002\, a^6 \big] \ , \end{aligned} \tag{10.17}$$

where $a = \alpha Z$ [28].

Although it is easy to use Eq. (10.16) to calculate X_0, the functional dependence on Z is somewhat hidden. Dahl provides a compact fit to the data [29]:

$$X_0 = \frac{716.4 \text{ g cm}^{-2}\, A}{Z(Z+1) \ln(287/\sqrt{Z})} \tag{10.18}$$