20. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A $\sqrt{\ }$ is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

J	J	…
M	M	…
m_1	m_2	
m_1	m_2	Coefficients
.	.	
.	.	
.	.	

1/2×1/2

		1	
		$+1$	1	0
	$+1/2$	$+1/2$	1	0	0
$+1/2$	$-1/2$	$1/2$	$1/2$	1
$-1/2$	$+1/2$	$1/2$	$-1/2$	-1
	$-1/2$	$-1/2$	1

1×1/2

		$3/2$		
		$+3/2$	$3/2$	$1/2$
	$+1$	$+1/2$	1	$+1/2$	$+1/2$
$+1$	$-1/2$	$1/3$	$2/3$	$3/2$	$1/2$
0	$+1/2$	$2/3$	$-1/3$	$-1/2$	$-1/2$
	0	$-1/2$	$2/3$	$1/3$	$3/2$
	-1	$+1/2$	$1/3$	$-2/3$	$-3/2$
		-1	$-1/2$	1

2×1

		3			
		$+3$	3	2	
	$+2$	$+1$	1	$+2$	$+2$	
$+2$	0	$1/3$	$2/3$	3	2	1
$+1$	$+1$	$2/3$	$-1/3$	$+1$	$+1$	$+1$
	$+2$	-1	$1/15$	$1/3$	$3/5$
	$+1$	0	$8/15$	$1/6$	$-3/10$
	0	$+1$	$6/15$	$-1/2$	$1/10$

1×1

		2			
		$+2$	2	1	
	$+1$	$+1$	1	$+1$	$+1$	
$+1$	0	$1/2$	$1/2$	2	1	0
0	$+1$	$1/2$	$-1/2$	0	0	0
	$+1$	-1	$1/6$	$1/2$	$1/3$	
	0	0	$2/3$	0	$-1/3$	
	-1	$+1$	$1/6$	$-1/2$	$1/3$	

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\,\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\,\sin^2\theta\, e^{2i\phi}$$

2×1/2

		$5/2$	
		$+5/2$	$5/2$	$3/2$
	$+2$	$1/2$	1	$3/2$	$+3/2$
$+2$	$-1/2$	$1/5$	$4/5$	$5/2$	$3/2$
$+1$	$+1/2$	$4/5$	$-1/5$	$+1/2$	$+1/2$
	$+1$	$-1/2$	$2/5$	$3/5$	$5/2$	$3/2$
	0	$+1/2$	$3/5$	$-2/5$	$-1/2$	$-1/2$
	0	$-1/2$	$3/5$	$2/5$	$5/2$	$3/2$
	-1	$+1/2$	$2/5$	$-3/5$	$-3/2$	$-3/2$
		-1	$-1/2$	$4/5$	$1/5$	$5/2$
		-2	$+1/2$	$1/5$	$-4/5$	$-5/2$
			-2	$-1/2$	1

3/2×1/2

		2			
		$+2$	2	1	
	$+3/2$	$+1/2$	1	$+1$	$+1$	
$+3/2$	$-1/2$	$1/4$	$3/4$	2	1	
$+1/2$	$+1/2$	$3/4$	$-1/4$	0	0	
	$+1/2$	$-1/2$	$1/2$	$1/2$	2	1
	$-1/2$	$+1/2$	$1/2$	$-1/2$	-1	-1
		$-1/2$	$-1/2$	$3/4$	$1/4$	2
		$-3/2$	$+1/2$	$1/4$	$-3/4$	-2
			$-3/2$	$-1/2$	1

3/2×1

		$5/2$			
		$+5/2$	$5/2$	$3/2$	
	$+3/2$	$+1$	1	$+3/2$	$+3/2$	
$+3/2$	0	$2/5$	$3/5$	$5/2$	$3/2$	$1/2$
$+1/2$	$+1$	$3/5$	$-2/5$	$+1/2$	$+1/2$	$+1/2$
	$+3/2$	-1	$1/10$	$2/5$	$1/2$	
	$+1/2$	0	$3/5$	$1/15$	$-1/3$	
	$-1/2$	$+1$	$3/10$	$-8/15$	$1/6$	
		$+1/2$	-1	$3/10$	$8/15$	$1/6$
		$-1/2$	0	$3/5$	$-1/15$	$-1/3$
		$-3/2$	$+1$	$1/10$	$-2/5$	$1/2$

		3	2	1
		0	0	0
	$+1$	-1	$1/5$	$1/2$	$3/10$
	0	0	$3/5$	0	$-2/5$
	-1	$+1$	$1/5$	$-1/2$	$3/10$

		3	2	1
		-1	-1	-1
	0	-1	$6/15$	$1/2$	$1/10$
	-1	0	$8/15$	$-1/6$	$-3/10$
	-2	$+1$	$1/15$	$-1/3$	$3/5$

		$5/2$	$3/2$	$1/2$
		$-1/2$	$-1/2$	$-1/2$
	$+1/2$	-1	$3/10$	$8/15$	$1/6$
	$-1/2$	0	$3/5$	$-1/15$	$-1/3$
	$-3/2$	$+1$	$1/10$	$-2/5$	$1/2$

		3	2
		-2	-2
	-1	-1	$2/3$	$1/3$
	-2	0	$1/3$	$-2/3$

		$5/2$	$3/2$
		$-3/2$	$-3/2$
	$-1/2$	-1	$3/5$	$2/5$
	$-3/2$	0	$2/5$	$-3/5$

		$5/2$	$3/2$
		$-3/2$	$-3/2$
	-1	-2	$4/5$	$1/5$
	-2	$+1/2$	$1/5$	$-4/5$

		$5/2$
		$-5/2$
	-2	$-1/2$	1

		2	1
		$+1$	$+1$
	$+1/2$	$-1/2$	$1/2$	$1/2$
	$-1/2$	$+1/2$	$1/2$	$-1/2$

		2	1
		-1	-1
	$-1/2$	$-1/2$	$3/4$	$1/4$
	$-3/2$	$+1/2$	$1/4$	$-3/4$

		2
		-2
	$-3/2$	$-1/2$	1

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$