$$p_{\rm cm} = p_{\rm beam}\frac{M}{E_{\rm cm}} \ . \tag{23.8}$$

For example, if a 0.80 GeV/c kaon beam is incident on a proton target, the center of mass energy is 1.699 GeV and the center of mass momentum of either particle is 0.442 GeV/c. It is also useful to note that

$$E_{\rm cm}\, dE_{\rm cm} = M\, dE_{\rm \ beam} = M\, \beta_{\rm \ beam}\, dp_{\rm \ beam} \ . \tag{23.9}$$

23.3. Lorentz-invariant amplitudes

The invariant amplitude $-i\mathscr{M}$ for a scattering or decay process is determined in perturbation theory by a set of Feynman diagrams. The convention of Bjorken and Drell is used except that fermion spinors are normalized so that $u\overline{u} = 2m$. As an example, the S-matrix for $2 \to 2$ scattering is related to $\mathscr{M}$ by

$$\begin{aligned}\langle p_1' p_2' \,|S|\, p_1 p_2\rangle &= I - i(2\pi)^4\, \delta^4(p_1 + p_2 - p_1' - p_2') \\ &\quad\times \frac{\mathscr{M}(p_1,\, p_2;\; p_1',\, p_2')}{(2E_1)^{1/2}\,(2E_2)^{1/2}\,(2E_1')^{1/2}\,(2E_2')^{1/2}} \ .\end{aligned} \tag{23.10}$$

The state normalization is such that

$$\langle p' | p\rangle = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{p}') \ . \tag{23.11}$$

23.4. Particle decays

The partial decay rate of a particle of mass M into n bodies in its rest frame is given in terms of the Lorentz-invariant matrix element $\mathscr{M}$ by

$$d\Gamma = \frac{(2\pi)^4}{2M}\, |\mathscr{M}|^2\, d\Phi_n\, (P;\, p_1,\, \ldots,\, p_n), \tag{23.12}$$

where $d\Phi_n$ is an element of n-body phase space given by

$$d\Phi_n(P;\, p_1,\, \ldots,\, p_n) = \delta^4\, (P - \sum_{i=1}^{n} p_i)\, \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2E_i} \ . \tag{23.13}$$

This phase space can be generated recursively, viz.

$$\begin{aligned}d\Phi_n(P;\, p_1,\, \ldots,\, p_n) &= d\Phi_j(q;\, p_1,\, \ldots,\, p_j) \\ &\quad\times\ d\Phi_{n-j+1}\, (P;\, q,\, p_{i+1},\, \ldots,\, p_n)(2\pi)^3 dq^2\ ,\end{aligned} \tag{23.14}$$

where $q^2 = (\sum_{i=1}^{j} E_i)^2 - \left|\sum_{i=1}^{j} \boldsymbol{p}_i\right|^2$. This form is particularly useful in the case where a particle decays into another particle which subsequently decays.